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Error estimates of local energy regularization for the logarithmic Schrödinger equation

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The logarithmic nonlinearity has been used in many partial differential equations (PDEs) for modeling problems in various applications. Due to the singularity of the logarithmic function, it introduces tremendous difficulties in establishing mathematical theories, as well as in designing and analyzing numerical methods for PDEs with such nonlinearity. Here, we take the logarithmic Schrödinger equation (LogSE) as a prototype model. Instead of regularizing $f(\rho) = \ln \rho$ in the LogSE directly and globally as being done in the literature, we propose a local energy regularization (LER) for the LogSE by first regularizing $F(\rho) = \rho \ln \rho - \rho$ locally near $\rho = 0^+$ with a polynomial approximation in

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the energy functional of the LogSE and then obtaining an energy regularized logarithmic Schrödinger equation (ERLogSE) via energy variation. Linear convergence is established between the solutions of ERLogSE and LogSE in terms of a small regularization parameter $0 < \varepsilon \ll 1$. Moreover, the conserved energy of the ERLogSE converges to that of LogSE quadratically, which significantly improves the linear convergence rate of the regularization method in the literature. Error estimates are also presented for solving the ERLogSE by using Lie–Trotter splitting integrators. Numerical results are reported to confirm our error estimates of the LER and of the time-splitting integrators for the ERLogSE. Finally, our results suggest that the LER performs better than regularizing the logarithmic nonlinearity in the LogSE directly.

Keywords: Logarithmic Schrödinger equation; logarithmic nonlinearity; energy regularization; error estimates; convergence rate; Lie–Trotter splitting.

AMS Subject Classification: 35Q40, 35Q55, 65M15, 81Q05

1. Introduction

The logarithmic nonlinearity appears in physical models from many fields. For example, the logarithmic nonlinearity is introduced in quantum mechanics or quantum optics, where a logarithmic Schrödinger equation (LogSE) is considered (e.g. Refs. 14–16 and 44),

$$i\partial_t u = -\Delta u + \lambda\, u \ln |u|^2, \quad \lambda \in \mathbb{R};$$

in oceanography and in fluid dynamics, with a logarithmic Korteweg–de Vries (KdV) equation or a logarithmic Kadomtsev–Petviashvili (KP) equation (e.g. Refs. 39,50 and 51); in quantum field theory and in inflation cosmology, via a logarithmic Klein–Gordon equation (e.g. Refs. 12,35 and 49); or in material sciences, by the introduction of a Cahn–Hilliard (CH) equation with logarithmic potentials (e.g. Refs. 24,28 and 33). Recently, the heat equation with a logarithmic nonlinearity has been investigated mathematically.[1,22]

In the context of quantum mechanics, the logarithmic nonlinearity was selected by assuming the separability of noninteracting subsystems property (cf. Ref. 14). This means that a solution of the nonlinear equation for the whole system can be constructed, as in the linear theory, by taking the product of two arbitrary solutions of the nonlinear equations for the subsystems. In other words, no correlations are introduced for noninteracting subsystems. As for the physical reality, robust physical grounds have been found for the application of equations with logarithmic nonlinearity. For instance, it was found in the stochastic formulation of quantum mechanics[45,48] that the logarithmic nonlinear term originates naturally from an internal stochastic force due to quantum fluctuations. Such kind of nonlinearity also appears naturally in inflation cosmology and in supersymmetric field theories.[11,30]

Remarkably enough for a nonlinear PDE, many explicit solutions are available for the logarithmic mechanics (see e.g. Refs. 14 and 43). For example, the logarithmic KdV equation, the logarithmic KP equation, the logarithmic Klein–Gordon equation give Gaussons: solitary wave solutions with Gaussian shapes.[50,51] In the case of LogSE (see Refs. 17 and 31), or the heat equation,[1] every initial Gaussian function evolves as a Gaussian: solving the corresponding nonlinear PDE is equivalent to solving ordinary differential equations (involving the purely time-dependent parameters of the Gaussian). However, we emphasize that this is not so in the case of, e.g. the logarithmic KdV equation, the logarithmic KP equation, or the logarithmic Klein–Gordon equation. This can be directly seen by trying to plug time-dependent Gaussian functions into these equations. Note that this distinction between various PDEs regarding the propagation of Gaussian functions is the same as at the linear level.

The well-posedness of the Cauchy problem for logarithmic equations is not trivial since the logarithmic nonlinearity is not locally Lipschitz continuous, due to the singularity of the logarithm at the origin. Existence was proved by compactness argument based on regularization of the nonlinearity, for the CH equation with a logarithmic potential[29] and the LogSE.[18] Uniqueness is also a challenging question, settled in the case of LogSE thanks to a surprising inequality discovered in Ref. 20, recalled in Lemma 2.1.

The singularity of the logarithmic nonlinearity also makes it very challenging to design and analyze numerical schemes. There have been extensive numerical works for the CH equation with a logarithmic Flory Huggins energy potential.[23,25,34,40,41,52] Specifically, a regularized energy functional was adopted for the CH equation with a logarithmic free energy.[25,52] A regularization of the logarithmic nonlinearity was introduced and analyzed in Refs. 4 and 5 in the case LogSE, see also Ref. 46.

In this paper, we introduce and analyze numerical methods for logarithmic equations via a local energy regularization (LER). We consider the LogSE as an example; the regularization can be extended to other logarithmic equations. The LogSE which arises in a model of nonlinear wave mechanics reads (cf. Ref. 14),

$$\begin{cases} i\partial_t u(\boldsymbol{x},t) = -\Delta u(\boldsymbol{x},t) + \lambda\, u(\boldsymbol{x},t)\, f(|u(\boldsymbol{x},t)|^2), & \boldsymbol{x}\in\Omega,\ \ t>0, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x}\in\overline{\Omega}, \end{cases} \tag{1.1}$$

where t and $\boldsymbol{x} \in \mathbb{R}^d$ $(d = 1,2,3)$ represent the temporal and spatial coordinates, respectively, $\lambda \in \mathbb{R}\backslash\{0\}$ measures the force of the nonlinear interaction, $u := u(\boldsymbol{x},t) \in \mathbb{C}$ is the dimensionless wave function, and

$$f(\rho) = \ln\rho, \quad \rho > 0, \quad \text{with } \rho = |u|^2. \tag{1.2}$$

The spatial domain is either $\Omega = \mathbb{R}^d$, or $\Omega \subset \mathbb{R}^d$ bounded with Lipschitz continuous boundary; in the latter case the equation is subject to homogeneous Dirichlet or periodic boundary conditions. This model has been widely applied in quantum mechanics, nuclear physics, geophysics, open quantum systems and Bose–Einstein condensation, see e.g. Refs. 3,26,37,38 and 53. We choose to consider positive time only merely to simplify the presentation, since (1.1) is time reversible. Formally, the flow of (1.1) enjoys two important conservations. The *mass*, defined as

$$N(t) := N(u(\cdot,t)) = \|u\|^2 = \int_\Omega |u(\boldsymbol{x},t)|^2 d\boldsymbol{x} \equiv N(u_0), \quad t \geq 0, \tag{1.3}$$

and the *energy*, defined as

$$\begin{aligned} E(t) : &= E(u(\cdot,t)) = \int_\Omega [|\nabla u(\boldsymbol{x},t)|^2 d\boldsymbol{x} + \lambda F(|u(\boldsymbol{x},t)|^2)] d\boldsymbol{x} \\ &\equiv \int_\Omega [|\nabla u_0(\boldsymbol{x})|^2 + \lambda F(|u_0(\boldsymbol{x})|^2)] d\boldsymbol{x} = E(u_0), \quad t \geq 0, \end{aligned} \tag{1.4}$$

where

$$F(\rho) = \int_0^\rho f(s) ds = \int_0^\rho \ln s \, ds = \rho \ln \rho - \rho, \quad \rho \geq 0. \tag{1.5}$$

The total angular momentum is also conserved, an identity that we do not use in the present paper. For the Cauchy problem (1.1) in a suitable functional framework, we refer to Refs. 17,20 and 36. For stability properties of standing waves for (1.1), we refer to Refs. 2,18 and 21. For the analysis of breathers and the existence of multisolitons, see Refs. 31 and 32.

In order to avoid numerical blow-up of the logarithmic nonlinearity at the origin, two models of regularized logarithmic Schrödinger equation (RLogSE) were proposed in Ref. 5, involving a direct regularization of f in (1.2), relying on a small regularized parameter $0 < \varepsilon \ll 1$,

$$\begin{cases} i\partial_t u^\varepsilon(\boldsymbol{x},t) = -\Delta u^\varepsilon(\boldsymbol{x},t) + \lambda\, u^\varepsilon(\boldsymbol{x},t)\, \widetilde{f}^\varepsilon(|u^\varepsilon(\boldsymbol{x},t)|)^2), & \boldsymbol{x} \in \Omega, \ t > 0, \\ u^\varepsilon(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \overline{\Omega}, \end{cases} \tag{1.6}$$

and

$$\begin{cases} i\partial_t u^\varepsilon(\boldsymbol{x},t) = -\Delta u^\varepsilon(\boldsymbol{x},t) + \lambda\, u^\varepsilon(\boldsymbol{x},t)\, \widehat{f}^\varepsilon(|u^\varepsilon(\boldsymbol{x},t)|^2)), & \boldsymbol{x} \in \Omega, \ t > 0, \\ u^\varepsilon(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \overline{\Omega}. \end{cases} \tag{1.7}$$

Here, $\widetilde{f}^\varepsilon(\rho)$ and $\widehat{f}^\varepsilon(\rho)$ are two types of regularization for $f(\rho)$, given by

$$\widetilde{f}^\varepsilon(\rho) = 2\ln(\varepsilon + \sqrt{\rho}), \quad \widehat{f}^\varepsilon(\rho) = \ln(\varepsilon^2 + \rho), \quad \rho \geq 0, \quad \text{with } \rho = |u^\varepsilon|^2. \tag{1.8}$$

Again, the RLogSEs (1.6) and (1.7) conserve the mass (1.3) with $u = u^\varepsilon$, as well as the *energies*

$$\widetilde{E}^\varepsilon(t) := \widetilde{E}^\varepsilon(u^\varepsilon(\cdot,t)) = \int_\Omega [|\nabla u^\varepsilon(\boldsymbol{x},t)|^2 d\boldsymbol{x} + \lambda \widetilde{F}^\varepsilon(|u^\varepsilon(\boldsymbol{x},t)|^2)] d\boldsymbol{x} \equiv \widetilde{E}^\varepsilon(u_0), \tag{1.9}$$

and

$$\widehat{E}^\varepsilon(t) := \widehat{E}^\varepsilon(u^\varepsilon(\cdot,t)) = \int_\Omega [|\nabla u^\varepsilon(\boldsymbol{x},t)|^2 d\boldsymbol{x} + \lambda \widehat{F}^\varepsilon(|u^\varepsilon(\boldsymbol{x},t)|^2)] d\boldsymbol{x} \equiv \widehat{E}^\varepsilon(u_0), \tag{1.10}$$

respectively, with, for $\rho \geq 0$,

$$\begin{aligned}\widetilde{F}^{\varepsilon}(\rho) &= \int_0^{\rho} \widetilde{f}^{\varepsilon}(s)ds = 2\rho\ln(\varepsilon+\sqrt{\rho}) + 2\varepsilon\sqrt{\rho} - \rho - 2\varepsilon^2\ln(1+\sqrt{\rho}/\varepsilon),\\ \widehat{F}^{\varepsilon}(\rho) &= \int_0^{\rho} \widehat{f}^{\varepsilon}(s)ds = (\varepsilon^2+\rho)\ln(\varepsilon^2+\rho) - \rho - 2\varepsilon^2\ln\varepsilon.\end{aligned} \tag{1.11}$$

The idea of this regularization is that the function $\rho \mapsto \ln\rho$ causes no (analytical or numerical) problem for large values of ρ, but is singular at $\rho = 0$. A linear convergence was established between the solutions of the LogSE (1.1) and the regularized model (1.6) or (1.7) for bounded Ω in terms of the small regularization parameter $0 < \varepsilon \ll 1$, i.e.

$$\sup_{t\in[0,T]} \|u^{\varepsilon}(t) - u(t)\|_{L^2(\Omega)} = O(\varepsilon), \quad \forall T > 0.$$

Applying this regularized model, a semi-implicit finite difference method (FDM) and a time-splitting method were proposed and analyzed for the LogSE (1.6) in Refs. 5 and 4, respectively. The above regularization saturates the nonlinearity in the region $\{\rho < \varepsilon^2\}$ (where $\rho = |u^{\varepsilon}|^2$), but of course has also some (smaller) effect in the other region $\{\rho > \varepsilon^2\}$, i.e. it regularizes $f(\rho) = \ln\rho$ globally.

Energy regularization is a method which has been adapted in different fields for dealing with singularity and/or roughness: in materials science, for establishing the well-posedness of the Cauchy problem for the CH equation with a logarithmic potential,[29] and for treating strongly anisotropic surface energy;[7,42] in mathematical physics, for the well-posedness of the LogSE;[18] in scientific computing, for designing regularized numerical methods in the presence of singularities.[9,25,52] The main goal of this paper is to present a LER for the LogSE (1.1). We regularize the interaction energy density $F(\rho)$ only locally in the region $\{\rho < \varepsilon^2\}$ by a sequence of polynomials, and keep it unchanged in $\{\rho > \varepsilon^2\}$. The choice of the regularized interaction energy density F_n^{ε} is prescribed by the regularity n imposed at this step, involving the matching conditions at $\{\rho = \varepsilon^2\}$. We then obtain a sequence of energy regularized logarithmic Schrödinger equations (ERLogSEs), from the regularized energy functional density F_n^{ε}, via energy variation. Unlike in Refs. 25 and 52, where the interaction energy density $F(\rho)$ is approximated by a second-order polynomial near the origin, here we present a systematic way to regularize the interaction energy density near the origin, i.e. locally, by a sequence of polynomials such that the order of regularity n of the overall regularized interaction energy density is arbitrary. We establish convergence rates between the solutions of ERLogSEs and LogSE in terms of the small regularized parameter $0 < \varepsilon \ll 1$. In addition, we also prove error estimates of numerical approximations of ERLogSEs by using time-splitting integrators.

The rest of this paper is organized as follows. In Sec. 2, we introduce a sequence of regularization F_n^{ε} for the logarithmic potential. A regularized model is derived and analyzed in Sec. 3 via the LER of the LogSE. Some numerical methods are proposed and analyzed in Sec. 4. In Sec. 5, we present numerical experiments. Throughout the paper, we adopt the standard L^2-based Sobolev spaces as well as the corresponding norms, and denote by C a generic positive constant independent of ε, the time step τ and the function u, and by $C(c)$ a generic positive constant depending on c.

2. Local Regularization for $F(\rho) = \rho \ln \rho - \rho$

We consider a local regularization starting from an approximation to the interaction energy density $F(\rho)$ in (1.5) (and thus in (1.4)).

2.1. *A sequence of local regularization*

In order to make a comparison with the former global regularization (1.6), we again distinguish the regions $\{\rho > \varepsilon^2\}$ and $\{\rho < \varepsilon^2\}$. Instead of saturating the nonlinearity in the second region, we regularize it locally as follows. For an arbitrary integer $n \geq 2$, we approximate $F(\rho)$ by a piecewise smooth function which is polynomial near the origin,

$$F_n^\varepsilon(\rho) = F(\rho)\chi_{\{\rho\geq\varepsilon^2\}} + P_{n+1}^\varepsilon(\rho)\chi_{\{\rho<\varepsilon^2\}}, \quad n \geq 2, \tag{2.1}$$

where $0 < \varepsilon \ll 1$ is a small regularization parameter, χ_A is the characteristic function of the set A, and P_{n+1}^ε is a polynomial of degree $n+1$. We demand $F_n^\varepsilon \in C^n([0, +\infty))$ and $F_n^\varepsilon(0) = F(0) = 0$ (this allows the regularized energy to be well-defined on the whole space). The above conditions determine P_{n+1}^ε, as we now check. Since $P_{n+1}^\varepsilon(0) = 0$, write

$$P_{n+1}^\varepsilon(\rho) = \rho\, Q_n^\varepsilon(\rho), \tag{2.2}$$

with Q_n^ε a polynomial of degree n. Correspondingly, denote $F(\rho) = \rho\, Q(\rho)$ with $Q(\rho) = \ln \rho - 1$. The continuity conditions read

$$P_{n+1}^\varepsilon(\varepsilon^2) = F(\varepsilon^2), \quad (P_{n+1}^\varepsilon)'(\varepsilon^2) = F'(\varepsilon^2), \ldots, \quad (P_{n+1}^\varepsilon)^{(n)}(\varepsilon^2) = F^{(n)}(\varepsilon^2),$$

which in turn yield

$$Q_n^\varepsilon(\varepsilon^2) = Q(\varepsilon^2), \quad (Q_n^\varepsilon)'(\varepsilon^2) = Q'(\varepsilon^2), \ldots, \quad (Q_n^\varepsilon)^{(n)}(\varepsilon^2) = Q^{(n)}(\varepsilon^2).$$

Thus Q_n^ε is nothing else but Taylor polynomial of Q of degree n at $\rho = \varepsilon^2$, i.e.

$$Q_n^\varepsilon(\rho) = Q(\varepsilon^2) + \sum_{k=1}^{n} \frac{Q^{(k)}(\varepsilon^2)}{k!}(\rho - \varepsilon^2)^k = \ln \varepsilon^2 - 1 - \sum_{k=1}^{n} \frac{1}{k}\left(1 - \frac{\rho}{\varepsilon^2}\right)^k. \tag{2.3}$$

In particular, Taylor's formula yields

$$Q(\rho) - Q_n^\varepsilon(\rho) = \int_{\varepsilon^2}^{\rho} Q^{(n+1)}(s)\frac{(\rho - s)^n}{n!}ds = \int_{\varepsilon^2}^{\rho} \frac{(s-\rho)^n}{s^{n+1}}ds. \tag{2.4}$$

Plugging (2.3) into (2.2), we get the explicit formula of $P^\varepsilon_{n+1}(\rho)$. We emphasize a formula which will be convenient for convergence results:

$$(Q^\varepsilon_n)'(\rho) = \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\left(1-\frac{\rho}{\varepsilon^2}\right)^{k-1} = \frac{1}{\rho}\left(1-\left(1-\frac{\rho}{\varepsilon^2}\right)^{n}\right), \quad 0 \le \rho \le \varepsilon^2. \tag{2.5}$$

2.2. *Properties of the local regularization functions*

Differentiating (2.1) with respect to ρ and noting (2.2), (2.3) and (2.5), we get

$$f^\varepsilon_n(\rho) = (F^\varepsilon_n)'(\rho) = \ln\rho\,\chi_{\{\rho\ge\varepsilon^2\}} + q^\varepsilon_n(\rho)\chi_{\{\rho<\varepsilon^2\}}, \quad \rho \ge 0, \tag{2.6}$$

where

$$\begin{aligned} q^\varepsilon_n(\rho) &= (P^\varepsilon_{n+1})'(\rho) = Q^\varepsilon_n(\rho) + \rho\,(Q^\varepsilon_n)'(\rho) \\ &= \ln(\varepsilon^2) - \frac{n+1}{n}\left(1-\frac{\rho}{\varepsilon^2}\right)^n - \sum_{k=1}^{n-1}\frac{1}{k}\left(1-\frac{\rho}{\varepsilon^2}\right)^k. \end{aligned}$$

Noticing that q^ε_n is increasing in $[0,\varepsilon^2]$, $\widetilde{f}^\varepsilon$ and $\widehat{f}^\varepsilon$ are increasing on $[0,\infty)$, thus all three types of regularization (2.1) and (1.11) preserve the convexity of F. Moreover, as a sequence of local regularization (or approximation) for the semi-smooth function $F(\rho) \in C^0([0,\infty)) \cap C^\infty((0,\infty))$, we have $F^\varepsilon_n \in C^n([0,+\infty))$ for $n \ge 2$, while $\widetilde{F}^\varepsilon \in C^1([0,\infty)) \cap C^\infty((0,\infty))$ and $\widehat{F}^\varepsilon \in C^\infty([0,\infty))$. Similarly, as a sequence of local regularization (or approximation) for the logarithmic function $f(\rho) = \ln\rho \in C^\infty((0,\infty))$, we observe that $f^\varepsilon_n \in C^{n-1}([0,\infty))$ for $n \ge 2$, while $\widehat{f}^\varepsilon \in C^\infty([0,\infty))$ and $\widetilde{f}^\varepsilon \in C^0([0,\infty)) \cap C^\infty((0,\infty))$.

Recall the following lemma, established initially in Lemma 1.1.1 of Ref. 20.

Lemma 2.1. *For $z_1, z_2 \in \mathbb{C}$, we have*

$$|\mathrm{Im}\big(\big(z_1\ln(|z_1|^2) - z_2\ln(|z_2|^2)\big)(\overline{z_1}-\overline{z_2})\big)| \le 2|z_1-z_2|^2,$$

where $\mathrm{Im}(z)$ *and* $\overline{z}$ *denote the imaginary part and the complex conjugate of* z, *respectively.*

Next, we highlight some properties of f^ε_n.

Lemma 2.2. *Let $n \ge 2$ and $\varepsilon > 0$. For z_1, $z_2 \in \mathbb{C}$, we have*

$$|f^\varepsilon_n(|z_1|^2) - f^\varepsilon_n(|z_2|^2)| \le \tfrac{4n|z_1-z_2|}{\max\{\varepsilon,\min\{|z_1|,|z_2|\}\}}, \tag{2.7}$$

$$|\mathrm{Im}[\big(z_1 f^\varepsilon_n(|z_1|^2) - z_2 f^\varepsilon_n(|z_2|^2)\big)(\overline{z_1}-\overline{z_2})]| \le 4n|z_1-z_2|^2, \tag{2.8}$$

$$|\rho(f^\varepsilon_n)'(\rho)| \le 3, \quad |\sqrt{\rho}(f^\varepsilon_n)'(\rho)| \le \tfrac{2n}{\varepsilon}, \quad |\rho^{3/2}(f^\varepsilon_n)''(\rho)| \le \tfrac{3n^2}{2\varepsilon}, \quad \rho \ge 0, \tag{2.9}$$

$$|f^\varepsilon_n(\rho)| \le \max\{|\ln A|, 2+\ln(n\varepsilon^{-2})\}, \quad \rho\in[0,A]. \tag{2.10}$$

Proof. When $|z_1|,|z_2| \ge \varepsilon$, we have

$$|f^\varepsilon_n(|z_1|^2) - f^\varepsilon_n(|z_2|^2)| = 2\ln\left(1 + \frac{||z_1|-|z_2||}{\min\{|z_1|,|z_2|\}}\right) \le \frac{2|z_1-z_2|}{\min\{|z_1|,|z_2|\}}.$$

A direct calculation gives

$$(f_n^\varepsilon)'(\rho) = \frac{1}{\rho}\chi_{\{\rho\geq\varepsilon^2\}} + \left(\frac{n}{\varepsilon^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-1} + \frac{1}{\varepsilon^2}\sum_{k=0}^{n-1}\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right)\chi_{\{\rho<\varepsilon^2\}}. \quad (2.11)$$

Thus when $|z_1| < |z_2| \leq \varepsilon$, we have

$$\begin{aligned}
|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)| &= \int_{|z_1|^2}^{|z_2|^2} (f_n^\varepsilon)'(\rho)d\rho \\
&= \frac{n}{\varepsilon^2}\int_{|z_1|^2}^{|z_2|^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-1}d\rho + \frac{1}{\varepsilon^2}\sum_{k=0}^{n-1}\int_{|z_1|^2}^{|z_2|^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^k d\rho \\
&\leq \frac{n}{\varepsilon^2}(|z_2|^2-|z_1|^2) + \frac{1}{\varepsilon^2}\sum_{k=0}^{n-1}(|z_2|^2-|z_1|^2) \\
&= \frac{2n}{\varepsilon^2}(|z_2|^2-|z_1|^2) \leq \frac{4n}{\varepsilon}|z_1-z_2|.
\end{aligned}$$

Another case when $|z_2| < |z_1| \leq \varepsilon$ can be established similarly. Supposing, for example, $|z_2| < \varepsilon < |z_1|$, denote by z_3 the intersection point of the circle $\{z \in \mathbb{C} : |z| = \varepsilon\}$ and the line segment connecting z_1 and z_2. Combining the inequalities above, we have

$$\begin{aligned}
|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)| &\leq |f_n^\varepsilon(|z_2|^2) - f_n^\varepsilon(|z_3|^2)| + |\ln(|z_1|^2) - \ln(|z_3|^2)| \\
&\leq \frac{4n}{\varepsilon}|z_2-z_3| + \frac{2}{\varepsilon}|z_1-z_3| \\
&\leq \frac{4n}{\varepsilon}(|z_2-z_3|+|z_1-z_3|) = \frac{4n}{\varepsilon}|z_1-z_2|,
\end{aligned}$$

which completes the proof for (2.7).

Noticing that

$$\begin{aligned}
&\mathrm{Im}[\left(z_1 f_n^\varepsilon(|z_1|^2) - z_2 f_n^\varepsilon(|z_2|^2)\right)(\overline{z_1}-\overline{z_2})] \\
&\quad = -\mathrm{Im}(\overline{z_1}z_2)f_n^\varepsilon(|z_2|^2) - \mathrm{Im}(z_1\overline{z_2})f_n^\varepsilon(|z_1|^2) \\
&\quad = \mathrm{Im}(\overline{z_1}z_2)[f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)] \\
&\quad = \frac{1}{2i}(\overline{z_1}z_2 - z_1\overline{z_2})[f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)],
\end{aligned}$$

and

$$\begin{aligned}
|\overline{z_1}z_2 - z_1\overline{z_2}| &= |z_2(\overline{z_1}-\overline{z_2}) + \overline{z_2}(z_2-z_1)| \leq 2|z_2|\,|z_1-z_2|, \\
|\overline{z_1}z_2 - z_1\overline{z_2}| &= |\overline{z_1}(z_2-z_1) + z_1(\overline{z_1}-\overline{z_2})| \leq 2|z_1|\,|z_1-z_2|,
\end{aligned}$$

which implies

$$|\overline{z_1}z_2 - z_1\overline{z_2}| \leq 2\min\{|z_1|,|z_2|\}\,|z_1-z_2|,$$

one can conclude (2.8) by applying (2.7).

It follows from (2.11) that

$$g(\rho) = \rho(f_n^\varepsilon)'(\rho) = \chi_{\{\rho\geq\varepsilon^2\}} + \left(\frac{n\rho}{\varepsilon^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-1} + \frac{\rho}{\varepsilon^2}\sum_{k=0}^{n-1}\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right)\chi_{\{\rho<\varepsilon^2\}}$$
$$= \chi_{\{\rho\geq\varepsilon^2\}} + \left(\frac{n\rho}{\varepsilon^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-1} + 1 - \left(1-\frac{\rho}{\varepsilon^2}\right)^n\right)\chi_{\{\rho<\varepsilon^2\}},$$

which gives that

$$g'(\rho)\chi_{\{\rho<\varepsilon^2\}} = \frac{n}{\varepsilon^2}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-2}\left[2-\frac{(n+1)\rho}{\varepsilon^2}\right].$$

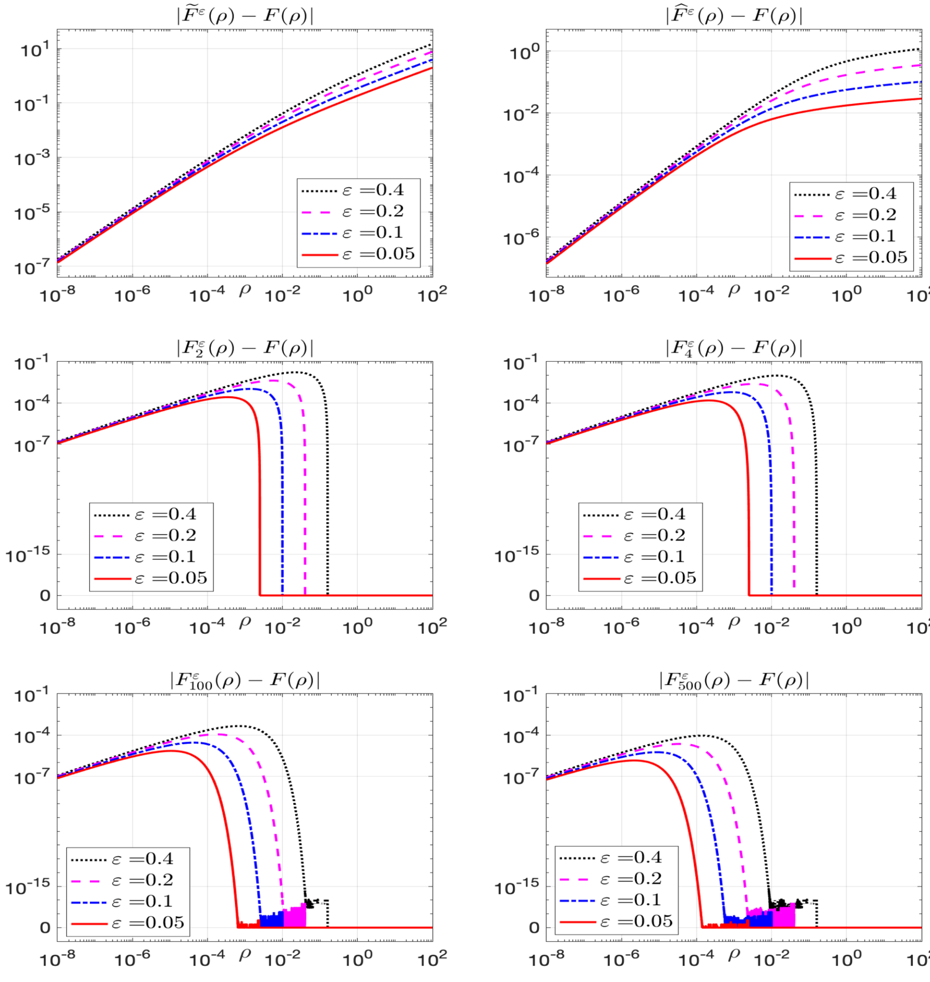

Fig. 1. (Color online) Comparison of different regularizations for $F(\rho) = \rho\ln\rho - \rho$.

This leads to

$$|\rho(f_n^\varepsilon)'(\rho)| = g(\rho) \le \max\left\{1, g\left(\frac{2\varepsilon^2}{n+1}\right)\right\} \le 1 + \frac{2n}{n+1} \le 3,$$

which completes the proof for the first inequality in (2.9). Finally, it follows from (2.11) that

$$\sqrt{\rho}(f_n^\varepsilon)'(\rho) = \frac{1}{\sqrt{\rho}}\chi_{\{\rho\ge\varepsilon^2\}} + \frac{\sqrt{\rho}}{\varepsilon^2}\left(n\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-1} + \sum_{k=0}^{n-1}\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right)\chi_{\{\rho<\varepsilon^2\}},$$

$$(f_n^\varepsilon)''(\rho) = -\frac{1}{\rho^2}\chi_{\{\rho\ge\varepsilon^2\}} - \left(\frac{n^2-1}{\varepsilon^4}\left(1-\frac{\rho}{\varepsilon^2}\right)^{n-2} + \frac{1}{\varepsilon^4}\sum_{k=0}^{n-3}(k+1)\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right) \times \chi_{\{\rho<\varepsilon^2\}},$$

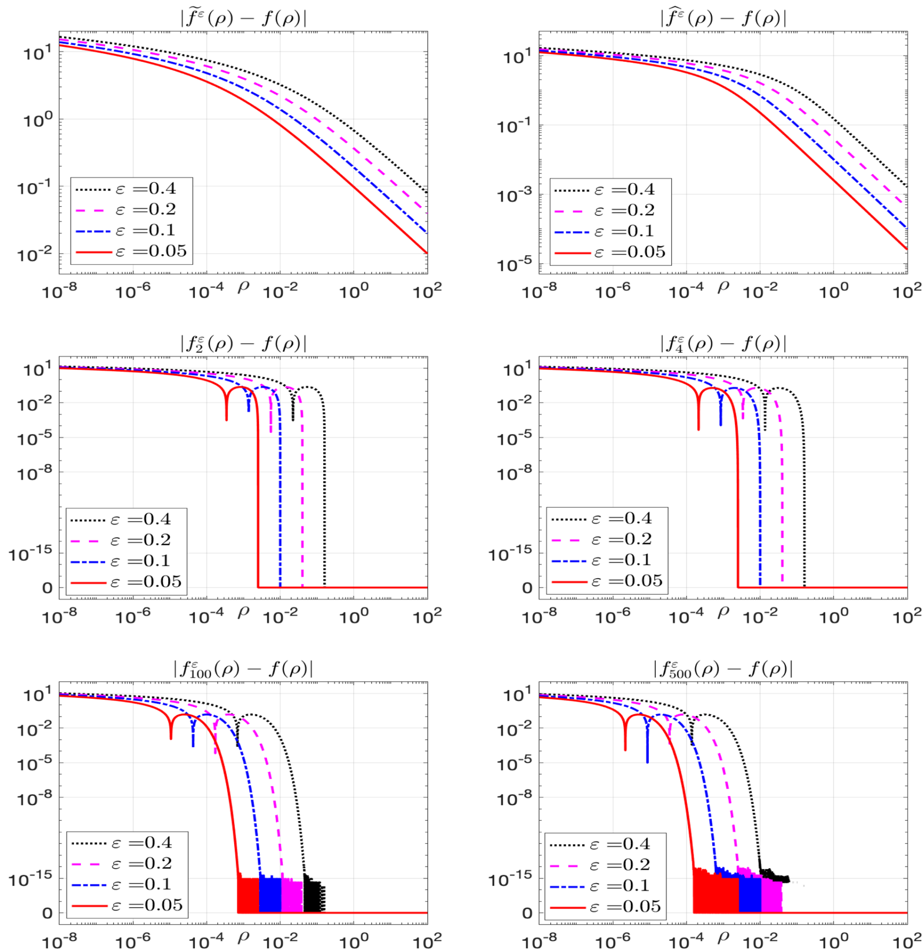

Fig. 2. (Color online) Comparison of different regularizations for the nonlinearity $f(\rho) = \ln\rho$.

which immediately yields that

$$
\begin{aligned}
&|\sqrt{\rho}(f_n^\varepsilon)'(\rho)| \le \frac{2n}{\varepsilon},\\
&|\rho^{3/2}(f_n^\varepsilon)''(\rho)| \le \frac{1}{\varepsilon}\left(n^2 - 1 + \sum_{k=0}^{n-3}(k+1)\right) = \frac{3n(n-1)}{2\varepsilon} < \frac{3n^2}{2\varepsilon}.
\end{aligned}
$$

For $\rho \in [0, \varepsilon^2]$, in view of $\varepsilon \in (0,1]$, one deduces

$$
|f_n^\varepsilon(\rho)| \le \ln(\varepsilon^{-2}) + \frac{n+1}{n}\left(1 - \frac{\rho}{\varepsilon^2}\right)^n + \sum_{k=1}^{n-1}\frac{1}{k}\left(1 - \frac{\rho}{\varepsilon^2}\right)^k
$$

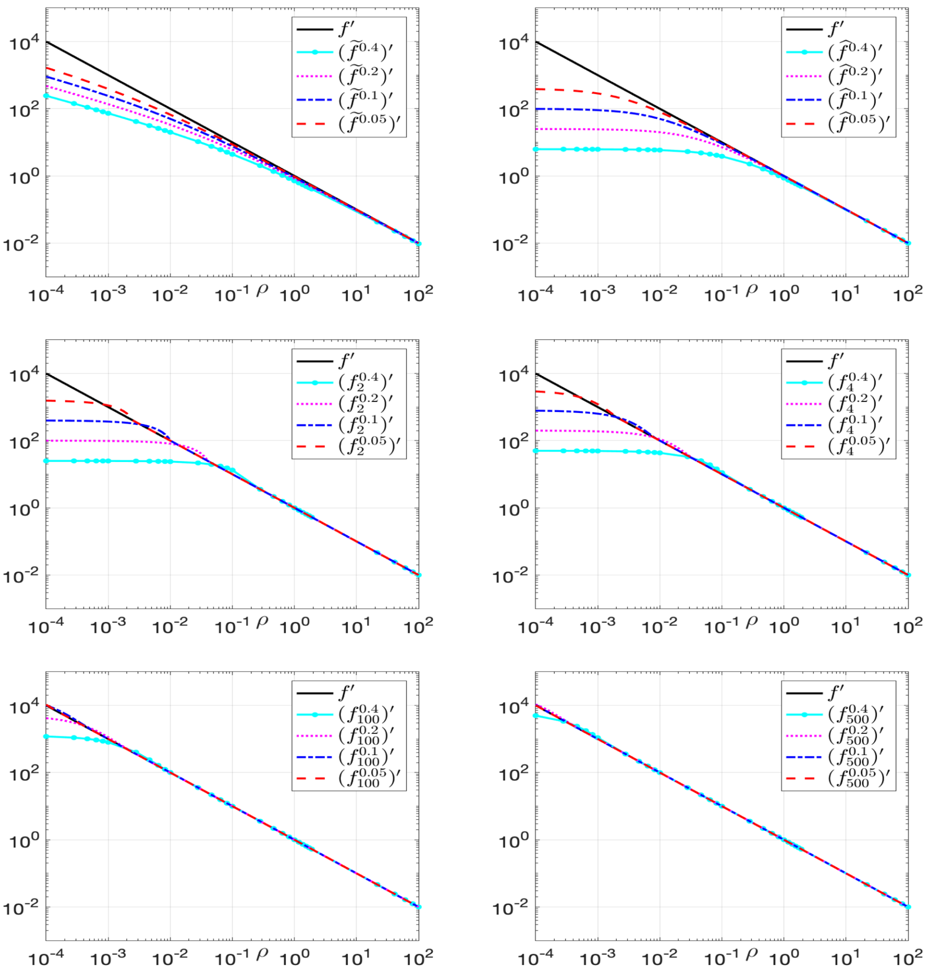

Fig. 3. (Color online) Comparison of different regularizations for $f'(\rho) = 1/\rho$.

$$\leq \ln(\varepsilon^{-2}) + \frac{n+1}{n} + \sum_{k=1}^{n-1} \frac{1}{k}$$

$$\leq \ln(\varepsilon^{-2}) + 2 + \sum_{k=2}^{n} \frac{1}{k}$$

$$\leq 2 + \ln(n\varepsilon^{-2}),$$

which together with $|f_n^\varepsilon(\rho)| \leq \max\{\ln(\varepsilon^{-2}), |\ln(A)|\}$ when $\rho \in [\varepsilon^2, A]$ concludes (2.10). □

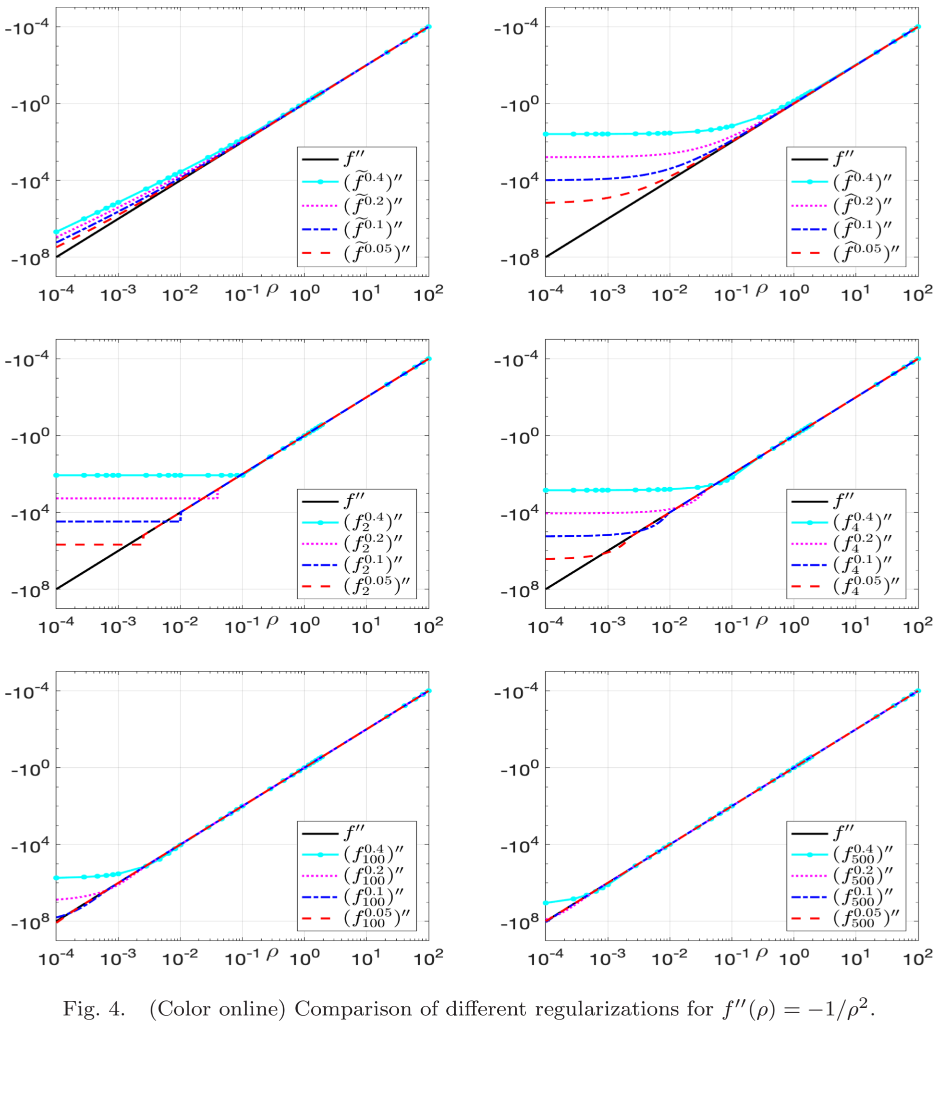

Fig. 4. (Color online) Comparison of different regularizations for $f''(\rho) = -1/\rho^2$.

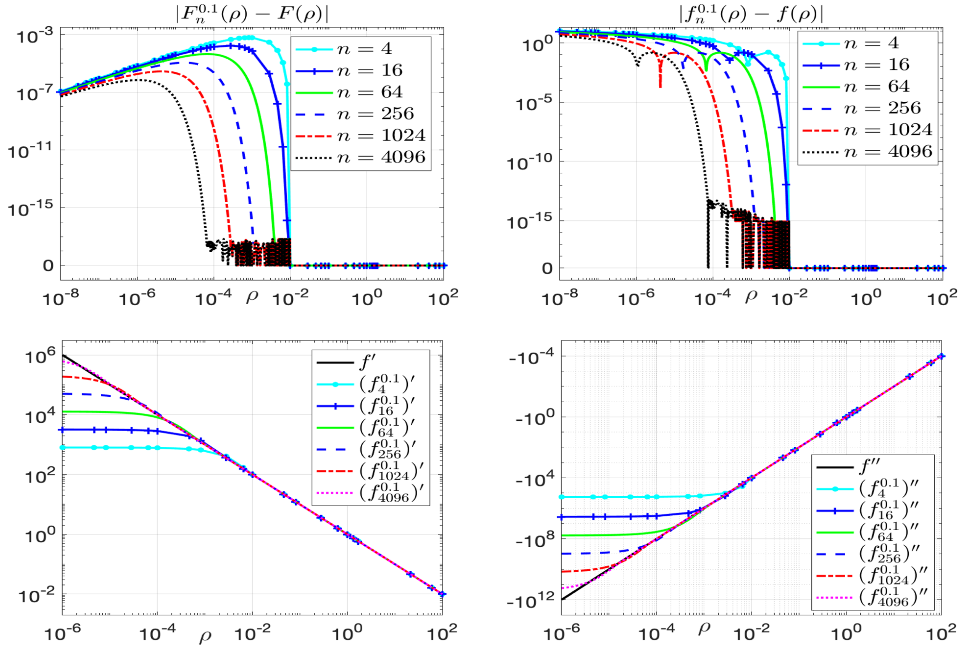

Fig. 5. (Color online) Comparison of regularizations $g_n^{0.1}$ ($g = F, f, f', f''$) with different order n.

2.3. *Comparison between different regularizations*

To compare different regularizations for $F(\rho)$ (and thus for $f(\rho)$), Fig. 1 shows F_n^ε ($n = 2, 4, 100, 500$), $\widetilde{F}^\varepsilon$ and $\widehat{F}^\varepsilon$ for different ε, from which we can see that the newly proposed local regularization F_n^ε approximates F more accurately.

Figure 2 shows various regularizations f_n^ε ($n = 2, 4, 100, 500$), $\widetilde{f}^\varepsilon$ and $\widehat{f}^\varepsilon$ for various ε, while Figs. 3 and 4 show their first- and second-order derivatives. From these figures, we can see that the newly proposed local regularization f_n^ε (and its derivatives with larger n) approximates the nonlinearity f (and its derivatives) more accurately. In addition, Fig. 5 depicts $F_n^\varepsilon(\rho)$ (with $\varepsilon = 0.1$) and its derivatives for different n, from which we can clearly see the convergence of $F_n^\varepsilon(\rho)$ (and its derivatives) to $F(\rho)$ (and its derivatives) with respect to order n.

3. Local Energy Regularization (LER) for the LogNLS

In this section, we consider the regularized energy

$$E_n^\varepsilon(u) := \int_\Omega [|\nabla u|^2 + \lambda F_n^\varepsilon(|u|^2)] d\boldsymbol{x}, \tag{3.1}$$

where F_n^ε is defined in (2.1). The Hamiltonian flow of the regularized energy $i\partial_t u = \frac{\delta E_n^\varepsilon(u)}{\delta \overline{u}}$ yields the following ERLogSE with a regularizing parameter $0 < \varepsilon \ll 1$,

$$\begin{cases} i\partial_t u^\varepsilon(\boldsymbol{x}, t) = -\Delta u^\varepsilon(\boldsymbol{x}, t) + \lambda\, u^\varepsilon(\boldsymbol{x}, t)\, f_n^\varepsilon(|u^\varepsilon(\boldsymbol{x}, t)|^2), & \boldsymbol{x} \in \Omega,\ \ t > 0, \\ u^\varepsilon(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \overline{\Omega}. \end{cases} \tag{3.2}$$

We recall that f_n^ε is defined by (2.6).

3.1. *The Cauchy problem*

To investigate the well-posedness of the problem (3.2), we first introduce some appropriate spaces. For $\alpha > 0$ and $\Omega = \mathbb{R}^d$, denote by L^2_α the weighted L^2 space

$$L^2_\alpha := \{v \in L^2(\mathbb{R}^d), \quad \boldsymbol{x} \mapsto \langle \boldsymbol{x} \rangle^\alpha v(\boldsymbol{x}) \in L^2(\mathbb{R}^d)\},$$

where $\langle \boldsymbol{x} \rangle := \sqrt{1 + |\boldsymbol{x}|^2}$, with norm $\|v\|_{L^2_\alpha} := \|\langle \boldsymbol{x} \rangle^\alpha v(\boldsymbol{x})\|_{L^2(\mathbb{R}^d)}$. Regarding the Cauchy problem (3.2), we have similar results as for the regularization (1.6) in Ref. 5, but not quite the same. For the convenience of the reader, we recall the main arguments.

Theorem 3.1. *Let $\lambda \in \mathbb{R}$, $u_0 \in H^1(\Omega)$, and $0 < \varepsilon \leq 1$.*

(1) *For* (3.2) *posed on $\Omega = \mathbb{R}^d$ or a bounded domain Ω with homogeneous Dirichlet or periodic boundary condition, there exists a unique, global weak solution $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1(\Omega))$ to* (3.2) (*with $H^1_0(\Omega)$ instead of $H^1(\Omega)$ in the Dirichlet case*). *Furthermore, for any given $T > 0$, there exists a positive constant $C(\lambda, T)$* (*independent of n*) *such that*

$$\|u^\varepsilon\|_{L^\infty([0,T]; H^1(\Omega))} \leq C(\lambda, T) \|u_0\|_{H^1(\Omega)}, \quad \forall\, \varepsilon > 0. \tag{3.3}$$

(2) *For* (3.2) *posed on a bounded domain Ω with homogeneous Dirichlet or periodic boundary condition, if in addition $u_0 \in H^2(\Omega)$, then $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2(\Omega))$ and there exists a positive constant $C(n, \lambda, T)$ such that*

$$\|u^\varepsilon\|_{L^\infty([0,T]; H^2(\Omega))} \leq C(n, \lambda, T) \|u_0\|_{H^2(\Omega)}, \quad \forall\, \varepsilon > 0. \tag{3.4}$$

(3) *For* (3.2) *on $\Omega = \mathbb{R}^d$, suppose moreover $u_0 \in L^2_\alpha$, for some $0 < \alpha \leq 1$.*

- *There exists a unique, global weak solution $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{R}^d) \cap L^2_\alpha)$ to* (3.2), *and*

$$\begin{aligned} &\|u^\varepsilon\|_{L^\infty([0,T]; H^1)} \leq C(n, \lambda, T) \|u_0\|_{H^1}, \\ &\|u^\varepsilon\|_{L^\infty([0,T]; L^2_\alpha)} \leq C(n, \lambda, T, \|u_0\|_{H^1}) \|u_0\|_{L^2_\alpha}, \quad \forall\, \varepsilon > 0. \end{aligned} \tag{3.5}$$

- *If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2(\mathbb{R}^d))$, and*

$$\|u^\varepsilon\|_{L^\infty([0,T]; H^2)} \leq C(n, \lambda, T, \|u_0\|_{H^2}, \|u_0\|_{L^2_\alpha}), \quad \forall\, \varepsilon > 0. \tag{3.6}$$

- *If $u_0 \in H^2(\mathbb{R}^d) \cap L^2_2$, then $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2(\mathbb{R}^d) \cap L^2_2)$.*

Proof. (1) For fixed $\varepsilon > 0$, the nonlinearity in (3.2) is locally Lipschitz continuous, and grows more slowly than any power of $|u^\varepsilon|$. Standard Cauchy theory for nonlinear Schrödinger equations implies that there exists a unique solution $u^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}; H^1(\Omega))$ to (3.2) (respectively, $u^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}; H^1_0(\Omega))$ in the Dirichlet case); see e.g. Corollary 3.3.11 and Theorem 3.4.1 in Ref. 19. In addition, the L^2-norm of u^ε is independent of time,

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}, \quad \forall\, t \in \mathbb{R}.$$

For $j \in \{1, \ldots, d\}$, differentiate (3.2) with respect to x_j:

$$(i\partial_t + \Delta)\, \partial_j u^\varepsilon = \lambda \partial_j u^\varepsilon f_n^\varepsilon(|u^\varepsilon|^2) + 2\lambda u^\varepsilon (f_n^\varepsilon)'(|u^\varepsilon|^2) \mathrm{Re}\,(\overline{u^\varepsilon}\partial_j u^\varepsilon)\,.$$

Multiply the above equation by $\partial_j \overline{u^\varepsilon}$, integrate on Ω, and take the imaginary part: (2.9) implies

$$\frac{1}{2}\frac{d}{dt}\|\partial_j u^\varepsilon\|^2_{L^2(\Omega)} \le 6|\lambda| \|\partial_j u^\varepsilon\|^2_{L^2(\Omega)},$$

hence (3.3), by Gronwall lemma.

(2) The propagation of the H^2 regularity is standard, since f_n^ε is smooth, so we focus on (3.4). We now differentiate (3.2) with respect to time: we get the same estimate as above, with ∂_j replaced by ∂_t, and so

$$\|\partial_t u^\varepsilon(t)\|^2_{L^2(\Omega)} \le \|\partial_t u^\varepsilon(0)\|^2_{L^2(\Omega)} e^{12|\lambda||t|}.$$

In view of (3.2),

$$i\partial_t u^\varepsilon_{|t=0} = -\Delta u_0 + \lambda u_0 f_n^\varepsilon(|u_0|^2).$$

For $0 < \delta < 1$, we have

$$\sqrt{\rho}|f_n^\varepsilon(\rho)| \le C(\delta)\left(\rho^{1/2-\delta/2} + \rho^{1/2+\delta/2}\right),$$

for some $C(\delta)$ independent of ε and n, so for $\delta > 0$ sufficiently small, Sobolev embedding entails

$$\|\partial_t u^\varepsilon(0)\|_{L^2(\Omega)} \le \|u_0\|_{H^2(\Omega)} + C(\delta)\left(\|u_0\|^{1-\delta}_{L^{2-2\delta}(\Omega)} + \|u_0\|^{1+\delta}_{H^1(\Omega)}\right).$$

Since Ω is bounded, Hölder inequality yields

$$\|u_0\|_{L^{2-2\delta}(\Omega)} \le \|u_0\|_{L^2(\Omega}|\Omega|^{\delta/(2-2\delta)}.$$

Thus, the first term in (3.2) is controlled in L^2. Using the same estimates as above, we control the last term in (3.2) (thanks to (3.3)), and we infer an L^2-estimate for Δu^ε, hence (3.4).

(3) In the case $\Omega = \mathbb{R}^d$, we multiply (3.2) by $\langle \boldsymbol{x}\rangle^\alpha$, and the same energy estimate as before now yields

$$\frac{d}{dt}\|u^\varepsilon\|^2_{L^2_\alpha} = 4\alpha\,\mathrm{Im}\int_{\mathbb{R}^d} \frac{\boldsymbol{x}\cdot\nabla u^\varepsilon}{\langle \boldsymbol{x}\rangle^{2-2\alpha}}\overline{u^\varepsilon}(t)\, d\boldsymbol{x} \lesssim \|\langle \boldsymbol{x}\rangle^{2\alpha-1} u^\varepsilon\|_{L^2(\mathbb{R}^d)}\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)}$$

$$\lesssim \|\langle \boldsymbol{x}\rangle^{\alpha} u^\varepsilon\|_{L^2(\mathbb{R}^d)}\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)},$$

where the last inequality follows from the assumption $\alpha \le 1$, hence (3.5). To prove (3.6), we resume the same approach as to get (3.4), with the difference that the Hölder estimate must be replaced by some other estimate (see e.g. Ref. 17): for $\delta > 0$ sufficiently small,

$$\int_{\mathbb{R}^d} |u|^{2-2\delta} \lesssim \|u\|_{L^2(\mathbb{R}^d)}^{2-2\delta-d\delta/\alpha} \||\boldsymbol{x}|^\alpha u\|_{L^2(\mathbb{R}^d)}^{d\delta/\alpha}.$$

The L^2_2 estimate follows easily, see e.g. Ref. 5 for details. □

3.2. *Convergence of the regularized model*

In this subsection, we show an approximation property of the regularized model (3.2) to (1.1).

Lemma 3.1. *Suppose Eq.* (3.2) *is set on* Ω, *where* $\Omega = \mathbb{R}^d$, *or* $\Omega \subset \mathbb{R}^d$ *is a bounded domain with homogeneous Dirichlet or periodic boundary condition. We have the general estimate*:

$$\frac{d}{dt}\|u^\varepsilon(t) - u(t)\|_{L^2}^2 \le |\lambda|(4\|u^\varepsilon(t) - u(t)\|_{L^2}^2 + 6\varepsilon\|u^\varepsilon(t) - u(t)\|_{L^1}). \tag{3.7}$$

Proof. Subtracting (1.1) from (3.2), we see that the error function $e^\varepsilon := u^\varepsilon - u$ satisfies

$$i\partial_t e^\varepsilon + \Delta e^\varepsilon = \lambda[u^\varepsilon \ln(|u^\varepsilon|^2) - u\ln(|u|^2)] + \lambda u^\varepsilon[f_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2)]\chi_{\{|u^\varepsilon|<\varepsilon\}}.$$

Multiplying the above error equation by $\overline{e^\varepsilon(t)}$, integrating in space and taking imaginary parts, we can get by using Lemma 2.1, (2.4) and (2.5) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e^\varepsilon(t)\|_{L^2}^2 &= 2\lambda\,\mathrm{Im}\int_\Omega [u^\varepsilon \ln(|u^\varepsilon|) - u\ln(|u|)]\overline{e^\varepsilon}(\boldsymbol{x},t)d\boldsymbol{x} \\
&\quad + \lambda\,\mathrm{Im}\int_{|u^\varepsilon|<\varepsilon} u^\varepsilon[f_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2)]\overline{e^\varepsilon}(\boldsymbol{x},t)d\boldsymbol{x} \\
&\le 2|\lambda|\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda|\Big|\int_{|u^\varepsilon|<\varepsilon} u^\varepsilon\overline{e^\varepsilon}[Q_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2) \\
&\quad + |u^\varepsilon|^2(Q_n^\varepsilon)'(|u^\varepsilon|^2)]d\boldsymbol{x}| \\
&\le 2|\lambda|\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda|\left|\int_{|u^\varepsilon|<\varepsilon} \overline{e^\varepsilon}u^\varepsilon\left[\int_{|u^\varepsilon|^2}^{\varepsilon^2} \frac{(s-|u^\varepsilon|^2)^n}{s^{n+1}}ds - 1\right.\right. \\
&\quad \left.\left. + |u^\varepsilon|^2(Q_n^\varepsilon)'(|u^\varepsilon|^2)\right] d\boldsymbol{x}\right|
\end{aligned}$$

$$= 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda| \left| \int_{|u^\varepsilon|<\varepsilon} \overline{e^\varepsilon} u^\varepsilon \left[\int_{|u^\varepsilon|^2}^{\varepsilon^2} \frac{(s-|u^\varepsilon|^2)^n}{s^{n+1}} ds \right.\right.$$

$$\left.\left. - \left(1 - \frac{|u^\varepsilon|^2}{\varepsilon^2}\right)^n \right] d\boldsymbol{x} \right|$$

$$\leq 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + \varepsilon|\lambda|\|e^\varepsilon\|_{L^1} + |\lambda| \left| \int_0^{\varepsilon^2} s^{-n-1} \int_{|u^\varepsilon|^2<s} \right.$$

$$\left. \times \overline{e^\varepsilon} u^\varepsilon (s-|u^\varepsilon|^2)^n d\boldsymbol{x} ds \right|$$

$$\leq 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + 3\varepsilon|\lambda|\|e^\varepsilon\|_{L^1}.$$

This yields the result. □

Invoking the same arguments as in Ref. 5, based on the previous error estimate, and interpolation between L^2 and H^2, we get the following error estimate.

Proposition 3.1. *If Ω has finite measure and $u_0 \in H^2(\Omega)$, then for any $T > 0$,*

$$\|u^\varepsilon - u\|_{L^\infty([0,T];L^2(\Omega))} \leq C_1\varepsilon, \quad \|u^\varepsilon - u\|_{L^\infty([0,T];H^1(\Omega))} \leq C_2\varepsilon^{1/2},$$

where C_1 depends on $|\lambda|$, T, $|\Omega|$, and C_2 depends in addition on $\|u_0\|_{H^2(\Omega)}$. If $\Omega = \mathbb{R}^d$, $1 \leq d \leq 3$ and $u_0 \in H^2(\mathbb{R}^d) \cap L_2^2$, then for any $T > 0$, we have

$$\|u^\varepsilon - u\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \leq D_1\varepsilon^{\frac{4}{4+d}}, \quad \|u^\varepsilon - u\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} \leq D_2\varepsilon^{\frac{2}{4+d}},$$

where D_1 and D_2 depend on d, $|\lambda|$, T, $\|u_0\|_{L_2^2}$ and $\|u_0\|_{H^2(\mathbb{R}^d)}$.

Proof. The proof is the same as that in Ref. 5. We just list the outline for the readers' convenience. When Ω is bounded, the convergence in L^2 follows from Gronwall's inequality by applying (3.7) and the estimate $\|v\|_{L^1} \leq |\Omega|^{1/2}\|v\|_{L^2}$. The estimate in H^1 follows form the Gagliardo–Nirenberg inequality $\|v\|_{H^1} \leq C\|v\|_{L^2}^{1/2}\|v\|_{H^2}^{1/2}$ and the property (3.4). For $\Omega = \mathbb{R}^d$, the convergence in L^2 can be established by Gronwall's inequality and the estimate (cf. Ref. 5)

$$\|v\|_{L^1} \leq C_d\|v\|_{L^2}^{1-d/4}\|v\|_{L_2^2}^{d/4} \leq C_d\left(\varepsilon^{-1}\|v\|_{L^2}^2 + \varepsilon^{\frac{4-d}{4+d}}\|v\|_{L_2^2}^{\frac{2d}{4+d}}\right),$$

which is derived by the Cauchy–Schwarz inequality and Young's inequality. The convergence in H^1 can similarly derived by the Gagliardo–Nirenberg inequality. □

3.3. *Convergence of the energy*

By construction, the energy is conserved, i.e.

$$E_n^\varepsilon(u^\varepsilon) = \int_\Omega [|\nabla u^\varepsilon(\boldsymbol{x},t)|^2 + \lambda F_n^\varepsilon(|u^\varepsilon(\boldsymbol{x},t)|^2)]d\boldsymbol{x} = E_n^\varepsilon(u_0). \tag{3.8}$$

For the convergence of the energy, we have the following estimate.

Proposition 3.2. *For $u_0 \in H^1(\Omega) \cap L^\alpha(\Omega)$ with $\alpha \in (0,2)$, the energy $E_n^\varepsilon(u_0)$ converges to $E(u_0)$ with*

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda|\, \|u_0\|_{L^\alpha}^\alpha \frac{\varepsilon^{2-\alpha}}{1-\alpha/2}.$$

In addition, for bounded Ω, we have

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda|\, |\Omega|\, \varepsilon^2.$$

Proof. It can be deduced from the definition (3.8) and (2.4) that

$$\begin{aligned}
|E_n^\varepsilon(u_0) - E(u_0)| &= |\lambda| \left| \int_\Omega [F(|u_0(\boldsymbol{x})|^2) - F_n^\varepsilon(|u_0(\boldsymbol{x})|^2)] d\boldsymbol{x} \right| \\
&= |\lambda| \left| \int_{|u_0(\boldsymbol{x})|<\varepsilon} |u_0(\boldsymbol{x})|^2 [Q(|u_0(\boldsymbol{x})|^2) - Q_n^\varepsilon(|u_0(\boldsymbol{x})|^2)] d\boldsymbol{x} \right| \\
&= |\lambda| \int_{|u_0(\boldsymbol{x})|<\varepsilon} |u_0(\boldsymbol{x})|^2 \int_{|u_0(\boldsymbol{x})|^2}^{\varepsilon^2} s^{-n-1}(s - |u_0(\boldsymbol{x})|^2)^n ds d\boldsymbol{x} \\
&= |\lambda| \int_0^{\varepsilon^2} s^{-n-1} \int_{|u_0(\boldsymbol{x})|^2<s} |u_0(\boldsymbol{x})|^2 (s - |u_0(\boldsymbol{x})|^2)^n d\boldsymbol{x} ds.
\end{aligned}$$

If Ω is bounded, we immediately get

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda|\, |\Omega|\, \varepsilon^2.$$

For unbounded Ω, one gets

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda| \int_0^{\varepsilon^2} s^{-n-1} s^{n+1-\alpha/2} \|u_0\|_{L^\alpha}^\alpha ds = |\lambda|\, \|u_0\|_{L^\alpha}^\alpha \frac{\varepsilon^{2-\alpha}}{1-\alpha/2},$$

which completes the proof. □

Remark 3.1. Recall that it was shown in Ref. 5 that for the regularized model (1.6) with the energy density (1.11), the energy

$$\widetilde{E}^\varepsilon(u_0) = \|\nabla u_0\|_{L^2}^2 + \lambda \int_\Omega \widetilde{F}^\varepsilon(|u_0|^2) d\boldsymbol{x} \tag{3.9}$$

converges to $E(u_0)$ with an error $O(\varepsilon)$. For the regularization (1.7) with the energy density (1.11) and the regularized energy

$$\widehat{E}^\varepsilon(u_0) = \|\nabla u_0\|_{L^2}^2 + \lambda \int_\Omega \widehat{F}^\varepsilon(|u_0|^2) d\boldsymbol{x}, \tag{3.10}$$

we have

$$\begin{aligned}
&|\widehat{E}^\varepsilon(u_0) - E(u_0)| \\
&\quad = |\lambda| \left| \int_\Omega [F(|u_0(\boldsymbol{x})|^2) - \widehat{F}^\varepsilon(|u_0(\boldsymbol{x})|^2)] d\boldsymbol{x} \right|
\end{aligned}$$

$$
\begin{aligned}
&= |\lambda| \left| \int_\Omega [(\varepsilon^2 + |u_0|^2)\ln(\varepsilon^2 + |u_0|^2) - \varepsilon^2 \ln(\varepsilon^2) - |u_0|^2 \ln(|u_0|^2)] d\boldsymbol{x} \right| \\
&\le |\lambda|\, \varepsilon^2 \int_\Omega \ln\left(1 + \frac{|u_0|^2}{\varepsilon^2}\right) d\boldsymbol{x} + |\lambda| \int_\Omega |u_0|^2 \ln\left(1 + \frac{\varepsilon^2}{|u_0|^2}\right) d\boldsymbol{x} \\
&\le |\lambda|\, \varepsilon^{2-\alpha} C(\alpha) \int_\Omega |u_0|^\alpha d\boldsymbol{x} \\
&= |\lambda|\, \varepsilon^{2-\alpha} C(\alpha) \|u_0\|_{L^\alpha}^\alpha,
\end{aligned}
$$

where we have used the inequality $\ln(1+x) \le C(\beta)x^\beta$ for $\beta \in (0,1]$ and $x \ge 0$. Hence for $u_0 \in H^1(\Omega) \cap L^\alpha(\Omega)$ with $\alpha \in (0,2)$, we infer

$$
|\widehat{E}^\varepsilon(u_0) - E(u_0)| \le |\lambda|\, \varepsilon^{2-\alpha} C(\alpha) \|u_0\|_{L^\alpha}^\alpha,
$$

that is, the same convergence rate as E_n^ε. Thus the newly proposed LER F_n^ε is more accurate than $\widetilde{F}^\varepsilon$, and than $\widehat{F}^\varepsilon$ in the case of bounded domains, from the viewpoint of energy.

4. Regularized Lie–Trotter Splitting Methods

In this section, we investigate approximation properties of the Lie–Trotter splitting (LTSP) methods[13,27,47] for solving the regularized model (3.2) in one dimension (1D). Extensions to higher dimensions are straightforward. To simplify notations, we set $\lambda = 1$.

4.1. *A time-splitting for* (3.2)

The operator splitting methods are based on a decomposition of the flow of (3.2):

$$
\partial_t u^\varepsilon = A(u^\varepsilon) + B(u^\varepsilon),
$$

where

$$
A(v) = i\Delta v, \quad B(v) = -iv f_n^\varepsilon(|v|^2),
$$

and the solution of the sub-equations

$$
\begin{cases} \partial_t v(x,t) = A(v(x,t)), & x \in \Omega,\ t > 0, \\ v(x,0) = v_0(x), \end{cases} \tag{4.1}
$$

$$
\begin{cases} \partial_t \omega(x,t) = B(\omega(x,t)), & x \in \Omega,\ t > 0, \\ \omega(x,0) = \omega_0(x), \end{cases} \tag{4.2}
$$

where $\Omega = \mathbb{R}$ or $\Omega \subset \mathbb{R}$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition on the boundary. Denote the flow of (4.1) and (4.2) by

$$
v(\cdot,t) = \Phi_A^t(v_0) = e^{it\Delta} v_0, \quad \omega(\cdot,t) = \Phi_B^t(\omega_0) = \omega_0 e^{-itf_n^\varepsilon(|\omega_0|^2)}, \ \ t \ge 0. \tag{4.3}
$$

As is well known, the flow Φ_A^t satisfies the isometry relation

$$\|\Phi_A^t(v_0)\|_{H^s} = \|v_0\|_{H^s}, \quad \forall\, s \in \mathbb{R}, \quad \forall\, t \geq 0. \tag{4.4}$$

Regarding the flow Φ_B^t, we have the following properties.

Lemma 4.1. *Assume $\tau > 0$ and $\omega_0 \in H^1(\Omega)$, then*

$$\|\Phi_B^\tau(\omega_0)\|_{L^2} = \|\omega_0\|_{L^2}, \quad \|\Phi_B^\tau(\omega_0)\|_{H^1} \leq (1 + 6\tau)\, \|\omega_0\|_{H^1}. \tag{4.5}$$

For v, $w \in L^2(\Omega)$,

$$\|\Phi_B^\tau(v) - \Phi_B^\tau(w)\|_{L^2} \leq (1 + 4n\tau)\, \|v - w\|_{L^2}. \tag{4.6}$$

Proof. By direct calculation, we get

$$\partial_x \Phi_B^\tau(\omega_0) = e^{-i\tau f_n^\varepsilon(|\omega_0|)^2}[\partial_x\omega_0 - i\tau(f_n^\varepsilon)'(|\omega_0|^2)(\omega_0^2\partial_x\overline{\omega_0} + |\omega_0|^2\partial_x\omega_0)],$$

which immediately gives (4.5) by recalling (2.9). We claim that for any $x \in \Omega$,

$$|\Phi_B^\tau(v)(x) - \Phi_B^\tau(w)(x)| \leq (1 + 4n\tau)\, |v(x) - w(x)|.$$

Assuming, for example, $|v(x)| \leq |w(x)|$, by inserting a term $v(x)e^{-i\tau f_n^\varepsilon(|w(x)|)^2}$, we can get

$$\begin{aligned}
&|\Phi_B^\tau(v)(x) - \Phi_B^\tau(w)(x)| \\
&\quad = |v(x)e^{-i\tau f_n^\varepsilon(|v(x)|)^2} - w(x)e^{-i\tau f_n^\varepsilon(|w(x)|)^2}| \\
&\quad = |v(x) - w(x) + v(x)(e^{i\tau[f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)]} - 1)| \\
&\quad \leq |v(x) - w(x)| + 2|v(x)| \left|\sin\left(\frac{\tau}{2}[f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)]\right)\right| \\
&\quad \leq |v(x) - w(x)| + \tau|v(x)|\, |f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)| \\
&\quad \leq (1 + 4n\tau)\, |v(x) - w(x)|,
\end{aligned}$$

where we have used the estimate (2.7). When $|v(x)| \geq |w(x)|$, the same inequality can be obtained by exchanging v and w in the above computation. Thus the proof for (4.6) is complete. □

4.2. *Error estimates for $\Phi^\tau = \Phi_A^\tau \Phi_B^\tau$*

We consider the LTSP

$$u^{\varepsilon,k+1} = \Phi^\tau(u^{\varepsilon,k}) = \Phi_A^\tau(\Phi_B^\tau(u^{\varepsilon,k})), \quad k \geq 0; \quad u^{\varepsilon,0} = u_0, \;\; \tau > 0. \tag{4.7}$$

For $u_0 \in H^1(\Omega)$, it follows from (4.4) and (4.5) that

$$\begin{aligned}
&\|u^{\varepsilon,k}\|_{L^2} = \|u^{\varepsilon,k-1}\|_{L^2} \equiv \|u^{\varepsilon,0}\|_{L^2} = \|u_0\|_{L^2}, \\
&\|u^{\varepsilon,k}\|_{H^1} \leq (1 + 6\tau)\, \|u^{\varepsilon,k-1}\|_{H^1} \leq e^{6k\tau}\|u_0\|_{H^1}, \quad k \geq 0.
\end{aligned} \tag{4.8}$$

Theorem 4.1. *Let $T > 0$ and $\tau_0 > 0$ be given constants. Assume that the solution of* (3.2) *satisfies $u^\varepsilon \in L^\infty([0,T]; H^1(\Omega))$ and the time step $\tau \le \tau_0$. Then there exists $0 < \varepsilon_0 < 1$ depending on n, τ_0 and $M := \|u^\varepsilon\|_{L^\infty([0,T];H^1(\Omega))}$ such that when $\varepsilon \le \varepsilon_0$ and $t_k := k\tau \le T$, we have*

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \le C(n, \tau_0, T, M) \ln(\varepsilon^{-1})\tau^{1/2}. \tag{4.9}$$

Proof. Denote the exact flow of (3.2) by $u^\varepsilon(t) = \Psi^t(u_0)$. First, we establish the local error for $v \in H^1(\Omega)$:

$$\|\Psi^\tau(v) - \Phi^\tau(v)\|_{L^2} \le C(n, \tau_0)\|v\|_{H^1} \ln(\varepsilon^{-1})\tau^{3/2}, \quad \tau \le \tau_0, \tag{4.10}$$

when ε is sufficiently small. Note that definitions imply

$$i\partial_t \Psi^t(v) + \Delta\Psi^t(v) = \Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2),$$
$$i\partial_t \Phi^t(v) + \Delta\Phi^t(v) = \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)).$$

Denote $\mathcal{E}^t(v) = \Psi^t(v) - \Phi^t(v)$, we have

$$i\partial_t \mathcal{E}^t(v) + \Delta\mathcal{E}^t(v) = \Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)). \tag{4.11}$$

Multiplying (4.11) by $\overline{\mathcal{E}^t(v)}$, integrating in space and taking the imaginary part, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2}^2 &= \mathrm{Im}(\Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)), \mathcal{E}^t(v)) \\
&= \mathrm{Im}(\Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2), \mathcal{E}^t(v)) \\
&\quad + \mathrm{Im}(\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)), \mathcal{E}^t(v)) \\
&\le 4n\|\mathcal{E}^t(v)\|_{L^2}^2 \\
&\quad + \|\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2))\|_{L^2}\|\mathcal{E}^t(v)\|_{L^2},
\end{aligned}$$

where we have used (2.8) and the scalar product is the standard one in L^2: $(u, w) = \int_\Omega u(x)\overline{w(x)}dx$. This implies

$$\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2} \le 4n\|\mathcal{E}^t(v)\|_{L^2} + J_1 + J_2, \tag{4.12}$$

where

$$J_1 = \|\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{L^2},$$
$$J_2 = \|\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2) - \Phi_A^t(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2))\|_{L^2}.$$

To estimate J_1 in (4.12), first we try to find the bound of $\|\Phi^t(v)\|_{L^\infty}, \|\Phi_B^t(v)\|_{L^\infty}$. It follows from (4.4) and (4.5) that

$$\|\Phi^t(v)\|_{H^1} = \|\Phi_B^t(v)\|_{H^1} \le (1 + 6t)\|v\|_{H^1} \le (1 + 6t_0)\|v\|_{H^1}, \quad t \le t_0.$$

Hence by Sobolev embedding, we have

$$\|\Phi^t(v)\|_{L^\infty} \le c(1+6t_0)\|v\|_{H^1}, \quad \|\Phi_B^t(v)\|_{L^\infty} \le c(1+6t_0)\|v\|_{H^1}, \tag{4.13}$$

where c is the constant in the Sobolev inequality $\|\omega\|_{L^\infty} \le c\|\omega\|_{H^1}$. Next, we claim that for y, z satisfying $|y|, |z| \le D$, it can be established that

$$|yf_n^\varepsilon(|y|^2) - zf_n^\varepsilon(|z|^2)| \le 4\ln(\varepsilon^{-1})|y-z|, \tag{4.14}$$

when ε is sufficiently small. It follows from (2.10) that $|f_n^\varepsilon(|y|^2)| \le 2 + \ln(n\varepsilon^{-2})$, when $|y| \le D$ and $\varepsilon \le \sqrt{n}/D$. Assuming, for example, $0 < |z| \le |y|$, and applying (2.7), we get

$$\begin{aligned}|yf_n^\varepsilon(|y|^2) - zf_n^\varepsilon(|z|^2)| &= |(y-z)f_n^\varepsilon(|y|^2)| + |z||f_n^\varepsilon(|y|^2) - f_n^\varepsilon(|z|^2)| \\ &\le (2+\ln(n\varepsilon^{-2}))|y-z| + |z|\frac{4n|y-z|}{|z|} \\ &\le 2(3n+\ln(\varepsilon^{-1}))|y-z| \\ &\le 4\ln(\varepsilon^{-1})|y-z|,\end{aligned}$$

when $\varepsilon \le \widetilde{\varepsilon} := \min\{\sqrt{n}/D, e^{-3n}\}$. The case when $y = 0$ or $z = 0$ can be handled similarly. Recalling (4.13), taking $D = c(1+6t_0)\|v\|_{H^1}$, we obtain, when $\varepsilon \le \varepsilon_1 := \min\{\frac{\sqrt{n}}{c(1+6t_0)\|v\|_{H^1}}, e^{-3n}\}$,

$$J_1 \le 4\ln(\varepsilon^{-1})\|\Phi^t(v) - \Phi_B^t(v)\|_{L^2} \le 4\ln(\varepsilon^{-1})\sqrt{2t}\|\Phi_B^t(v)\|_{H^1} \le 6\ln(\varepsilon^{-1})\sqrt{t}\|v\|_{H^1}, \tag{4.15}$$

where we have used the estimate

$$\|\omega - \Phi_A^t(\omega)\|_{L^2} \le \sqrt{2t}\,\|\omega\|_{H^1}, \tag{4.16}$$

as in Ref. 4, instead of the estimate from Ref. 13,

$$\|\omega - \Phi_A^t(\omega)\|_{L^2} \le 2t\|\omega\|_{H^2},$$

which in our case yields an extra $1/\varepsilon$ factor in the error estimate.

To estimate J_2, we first claim that

$$\|\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{H^1} \le 6\ln(\varepsilon^{-1})(1+3t_0)\|v\|_{H^1}, \tag{4.17}$$

when $\varepsilon \le \varepsilon_1$ and $t \le t_0$. Recalling that

$$\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2) = vf_n^\varepsilon(|v|^2)e^{-itf_n^\varepsilon(|v|^2)},$$

and $|f_n^\varepsilon(|v|^2)| \le 3\ln(\varepsilon^{-1})$, when $\varepsilon \le \varepsilon_1$, this implies

$$\|(\Phi_B^t(v))f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{L^2} \le 3\ln(\varepsilon^{-1})\|v\|_{L^2}.$$

Noticing that

$$\begin{aligned}\partial_x[\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)] &= e^{-itf_n^\varepsilon(|v|^2)}[v_xf_n^\varepsilon(|v|^2) \\ &\quad + (1-itf_n^\varepsilon(|v|^2))(f_n^\varepsilon)'(|v|^2)(v^2\overline{v_x} + |v|^2v_x)],\end{aligned}$$

which together with (2.9) yields

$$|\partial_x[\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)]| \leq [6+3\ln(\varepsilon^{-1})(1+6t_0)]|v_x| \leq 6\ln(\varepsilon^{-1})(1+3t_0)|v_x|,$$

which immediately gives (4.17). Applying (4.16) again entails

$$J_2 \leq \sqrt{2t}\,\|(\Phi_B^t(v))f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{H^1} \leq 9\ln(\varepsilon^{-1})(1+3t_0)\sqrt{t}\|v\|_{H^1}, \tag{4.18}$$

for $\varepsilon \leq \varepsilon_1$ and $t \leq t_0$. Combining (4.12), (4.15) and (4.18), we get

$$\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2} \leq 4n\|\mathcal{E}^t(v)\|_{L^2} + 15(1+2t_0)\ln(\varepsilon^{-1}))\sqrt{t}\|v\|_{H^1}.$$

Invoking Gronwall's inequality, we have

$$\begin{aligned}\|\mathcal{E}^\tau(v)\|_{L^2} &\leq e^{4n\tau}[\|\mathcal{E}^0(v)\|_{L^2} + 15(1+2\tau_0)\ln(\varepsilon^{-1})\|v\|_{H^1}\int_0^\tau \sqrt{s}ds]\\ &\leq 30(1+2\tau_0)e^{4n\tau}\|v\|_{H^1}\ln(\varepsilon^{-1})\tau^{3/2}\\ &\leq C(n,\tau_0)\|v\|_{H^1}\ln(\varepsilon^{-1})\tau^{3/2},\end{aligned}$$

when $\tau \leq \tau_0$ and $\varepsilon \leq \varepsilon_0 := \min\{\frac{\sqrt{n}}{c(1+6\tau_0)M}, e^{-3n}\}$ depending on τ_0, n and $M = \|u^\varepsilon\|_{L^\infty([0,T];H^1)}$, which completes the proof for (4.10).

Next, we infer the stability analysis for the operator Φ^t:

$$\|\Phi^\tau(v) - \Phi^\tau(w)\|_{L^2} \leq (1+4n\tau)\|v-w\|_{L^2}, \quad \text{for} \quad v,w \in L^2(\Omega). \tag{4.19}$$

Noticing that Φ_A^τ is a linear isometry on $H^s(\Omega)$, (4.6) gives (4.19) directly. Thus the error (4.9) can be established by combining the local error (4.10), the stability property (4.19) and a standard argument:[4,13]

$$\begin{aligned}&\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2}\\ &\quad = \|\Phi^\tau(u^{\varepsilon,k-1}) - \Psi^\tau(u^\varepsilon(t_{k-1})\|_{L^2}\\ &\quad \leq \|\Phi^\tau(u^{\varepsilon,k-1}) - \Phi^\tau(u^\varepsilon(t_{k-1}))\|_{L^2} + \|\Phi^\tau(u^\varepsilon(t_{k-1})) - \Psi^\tau(u^\varepsilon(t_{k-1}))\|_{L^2}\\ &\quad \leq (1+4n\tau)\|u^{\varepsilon,k-1} - u^\varepsilon(t_{k-1})\|_{L^2} + C(n,\tau_0)\ln(\varepsilon^{-1})\tau^{3/2}\|u^\varepsilon(t_{k-1})\|_{H^1}\\ &\quad \leq (1+4n\tau)\|u^{\varepsilon,k-1} - u^\varepsilon(t_{k-1})\|_{L^2} + MC(n,\tau_0)\ln(\varepsilon^{-1})\tau^{3/2}\\ &\quad \leq (1+4n\tau)^2\|u^{\varepsilon,k-2} - u^\varepsilon(t_{k-2})\|_{L^2} + MC(n,\tau_0)\ln(\varepsilon^{-1})\tau^{3/2}[1+(1+4n\tau)]\\ &\quad \leq \cdots\\ &\quad \leq (1+4n\tau)^k\|u^{\varepsilon,0} - u_0\|_{L^2} + MC(n,\tau_0)\ln(\varepsilon^{-1})\tau^{3/2}\sum_{j=0}^{k-1}(1+4n\tau)^j\\ &\quad \leq C(n,\tau_0,T,M)\ln(\varepsilon^{-1})\tau^{1/2},\end{aligned}$$

which completes the proof. □

Remark 4.1. As established in Theorem 3.1, for an arbitrarily large fixed $T > 0$, we have $u^\varepsilon \in L^\infty([0,T]; H^1(\Omega))$ as soon as $u_0 \in H^1(\Omega)$ when Ω is bounded. More specifically,

$$M = \|u^\varepsilon\|_{L^\infty([0,T];H^j)} \le C(n, \lambda, T, \|u_0\|_{H^j}), \quad j = 1, 2,$$

for a constant C independent of ε. When $\Omega = \mathbb{R}^d$, we require in addition $u_0 \in L^2_\alpha$ for some $0 < \alpha \le 1$ and C depends additionally on $\|u_0\|_{L^2_\alpha}$. Hence the constant in (4.9) as well as (4.22) in Theorem 4.2 is independent of ε.

Remark 4.2. By applying similar arguments as in Ref. 4, for $d = 2, 3$, the error estimate (4.9) can be established under a more restrictive condition $u^\varepsilon \in L^\infty([0,T]; H^2(\Omega))$, in which case ε_0 depends on n and $\|u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))}$, and $\|\Phi_B^t(v)\|_{H^2}$ has to be further investigated due to the Sobolev inequality $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. For details, we refer to Ref. 4.

4.3. *Error estimates for* $\Phi^\tau = \Phi_B^\tau \Phi_A^\tau$

We consider another LTSP

$$u^{\varepsilon,k+1} = \Phi^\tau(u^{\varepsilon,k}) = \Phi_B^\tau(\Phi_A^\tau(u^{\varepsilon,k})), \quad k \ge 0; \;\; u^{\varepsilon,0} = u_0, \;\tau \in (0, \tau_0]. \tag{4.20}$$

In the same fashion as above, we have

$$\|u^{\varepsilon,k}\|_{L^2} = \|u_0\|_{L^2}, \quad \|u^{\varepsilon,k}\|_{H^1} \le e^{6k\tau}\|u_0\|_{H^1}, \;\; k \ge 0. \tag{4.21}$$

Theorem 4.2. *Let $T > 0$. Assume that the solution of (3.2) satisfies $u^\varepsilon \in L^\infty([0,T]; H^2(\Omega))$. Then there exists $\varepsilon_0 > 0$ depending on n, τ_0 and $M = \|u^\varepsilon\|_{L^\infty([0,T];H^1(\Omega))}$ such that when $\varepsilon \le \varepsilon_0$ and $k\tau \le T$, we have*

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \le C(n, \tau_0, T, \|u^\varepsilon\|_{L^\infty([0,T];H^2(\Omega))})\frac{\tau}{\varepsilon}, \tag{4.22}$$

where $C(\cdot,\cdot,\cdot,\cdot)$ is independent of ε.

Proof. First, we prove the local error estimate: for $v_0 \in H^1(\Omega)$,

$$\|\Psi^\tau(v_0) - \Phi^\tau(v_0)\|_{L^2} \le C(n, \|v_0\|_{H^2})\frac{\tau^2}{\varepsilon}, \quad \varepsilon \le \widetilde{\varepsilon}_0, \tag{4.23}$$

where $\Phi^\tau = \Phi_B^\tau\Phi_A^\tau$, $\Psi^\tau(v_0)$ is the exact flow of (3.2) with initial data v_0 and $C(\cdot, \alpha)$ is increasing with respect to α and $\widetilde{\varepsilon}_0$ depends on n and $\|v_0\|_{H^1}$. We start from the Duhamel formula for $v(t) = \Psi^t(v_0)$:

$$\Psi^t(v_0) = e^{it\Delta}v_0 + \int_0^t e^{i(t-s)\Delta}B(v(s))ds. \tag{4.24}$$

Recall

$$B(v(s)) = B(e^{is\Delta}v_0) + \int_0^s dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]dy, \tag{4.25}$$

which is the variation-of-constants formula

$$B(g(s)) - B(g(0)) = \int_0^s dB(g(y))[g'(y)]dy, \quad g(y) = e^{i(s-y)\Delta}v(y).$$

Here $dB(\cdot)[\cdot]$ is the Gâteaux derivative:

$$\begin{aligned} dB(w_1)[w_2] &= \lim_{\delta\to 0} \frac{B(w_1+\delta w_2) - B(w_1)}{\delta} \\ &= -iw_2 f_n^\varepsilon(|w_1|^2) - iw_1(f_n^\varepsilon)'(|w_1|^2)[w_1\overline{w_2} + \overline{w_1}w_2]. \end{aligned} \tag{4.26}$$

Plugging (4.25) into (4.24) with $t = \tau$, we get

$$\Psi^\tau(v_0) = e^{i\tau\Delta}v_0 + \int_0^\tau e^{i(\tau-s)\Delta}B(e^{is\Delta}v_0)ds + e_1,$$

where

$$e_1 = \int_0^\tau \int_0^s e^{i(\tau-s)\Delta} dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]dyds.$$

On the other hand, for the Lie splitting $\Phi^\tau(v_0) = \Phi_B^\tau \Phi_A^\tau(v_0)$, applying the first-order Taylor expansion

$$\Phi_B^\tau(w) = w + \tau B(w) + \tau^2 \int_0^1 (1-s)dB(\Phi_B^{s\tau}(w))[B(\Phi_B^{s\tau}(w))]ds,$$

for $w = \Phi_A^\tau(v_0) = e^{i\tau\Delta}v_0$, we get

$$\Phi^\tau(v_0) = \Phi_B^\tau\Phi_A^\tau(v_0) = e^{i\tau\Delta}v_0 + \tau B(e^{i\tau\Delta}v_0) + e_2,$$

with

$$e_2 = \tau^2 \int_0^1 (1-s)dB(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))[B(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))]ds.$$

Thus

$$\Psi^\tau(v_0) - \Phi^\tau(v_0) = e_1 - e_2 + e_3,$$

where

$$e_3 = \int_0^\tau e^{i(\tau-s)\Delta}B(e^{is\Delta}v_0)ds - \tau B(e^{i\tau\Delta}v_0).$$

Noticing that e_3 is the quadrature error of the rectangle rule approximating the integral on $[0,\tau]$ of the function $g(s) = e^{i(\tau-s)\Delta}B(e^{is\Delta}v_0)$, this implies

$$e_3 = -\tau^2 \int_0^1 \theta g'(\theta\tau)d\theta,$$

where $g'(s) = -e^{i(\tau-s)\Delta}[A,B](e^{is\Delta}v_0)$, with

$$\begin{aligned} [A,B](w) &= dA(w)[Bw] - dB(w)[Aw] = i\Delta(Bw) - dB(w)[Aw] \\ &= (f_n^\varepsilon)'(|w|^2)(2w_x^2\overline{w} + 4w|w_x|^2 + 3w^2\overline{w_{xx}} - |w|^2 w_{xx}) \\ &\quad + w(f_n^\varepsilon)''(|w|^2)(w_x\overline{w} + w\overline{w_x})^2, \end{aligned}$$

by recalling (4.26) and

$$dA(w_1)[w_2] = \lim_{\delta\to 0} \frac{A(w_1+\delta w_2) - A(w_1)}{\delta} = i\Delta w_2. \tag{4.27}$$

Applying (2.9), we get

$$|[A,B](w)| \le \frac{12n+6n^2}{\varepsilon}|w_x|^2 + 12|w_{xx}|,$$

which implies

$$\begin{aligned}\|[A,B](w)\|_{L^2} &\le \frac{12n+6n^2}{\varepsilon}\|w_x\|_{L^4}^2 + 12\|w_{xx}\|_{L^2}\\ &\le \frac{12n+6n^2}{\varepsilon}\|w_x\|_{L^\infty}\|w_x\|_{L^2} + 12\|w_{xx}\|_{L^2}\\ &\le 12\|w\|_{H^2} + \frac{12cn^2}{\varepsilon}\|w\|_{H^2}^2,\end{aligned}$$

where we have used $n \ge 2$ and the Sobolev embedding $\|w\|_{L^\infty} \le c\|w\|_{H^1}$ for $d = 1$. This yields that for any $s \in [0,1]$,

$$\|g'(s)\|_{L^2} = \|[A,B](e^{is\Delta}v_0)\|_{L^2} \le 12\|v_0\|_{H^2}(1 + cn^2\|v_0\|_{H^2}/\varepsilon),$$

which immediately gives

$$\|e_3\|_{L^2} \le \tau^2 \int_0^1 \|g'(\theta\tau)\|_{L^2}d\theta \le 12\|v_0\|_{H^2}(1 + cn^2\|v_0\|_{H^2}/\varepsilon)\tau^2. \tag{4.28}$$

Next, we estimate e_1 and e_2. In view of (2.9), we have

$$\|dB(w_1)[w_2]\|_{L^2} \le (8 + \ln(n\varepsilon^{-2}))\|w_2\|_{L^2},$$

when $\varepsilon \le \widetilde{\varepsilon} := \sqrt{n}/\|w_1\|_{L^\infty}$. Thus one gets

$$\begin{aligned}\|dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]\|_{L^2} &\le (8 + \ln(n\varepsilon^{-2}))\|e^{i(s-y)\Delta}B(v(y))\|_{L^2}\\ &= (8 + \ln(n\varepsilon^{-2}))\|B(v(y))\|_{L^2},\end{aligned}$$

when $\varepsilon \le \varepsilon_1 = \sqrt{n}/\|e^{i(s-y)\Delta}v(y)\|_{L^\infty}$. By Sobolev embedding,

$$\|e^{i(s-y)\Delta}v(y)\|_{L^\infty} \le c\|e^{i(s-y)\Delta}v(y)\|_{H^1} = c\|\Psi^y(v_0)\|_{H^1}, \tag{4.29}$$

thus when $\varepsilon \le \varepsilon_2 := \frac{\sqrt{n}/c}{\max_{y\in[0,\tau]}\|\Psi^y(v_0)\|_{H^1}}$, we have

$$\begin{aligned}\|e_1\|_{L^2} &\le \int_0^\tau \int_0^s \|dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]\|_{L^2}dyds\\ &\le (8 + \ln(n\varepsilon^{-2}))\int_0^\tau\int_0^s \|B(v(y))\|_{L^2}dyds\\ &\le (8 + \ln(n\varepsilon^{-2}))\tau^2 \max_{0\le y\le\tau}\|v(y)f_n^\varepsilon(|v(y)|^2)\|_{L^2}\\ &\le (8 + \ln(n\varepsilon^{-2}))^2\tau^2 \max_{0\le y\le\tau}\|v(y)\|_{L^2}\\ &= (8 + \ln(n\varepsilon^{-2}))^2\|v_0\|_{L^2}\tau^2.\end{aligned} \tag{4.30}$$

Similarly, by recalling

$$\|\Phi_B^{s\tau}(e^{i\tau\Delta}v_0)\|_{L^\infty} = \|e^{i\tau\Delta}v_0\|_{L^\infty} \le c\|v_0\|_{H^1},$$

when $\varepsilon \le \varepsilon_3 := \sqrt{n}/(c\|v_0\|_{H^1})$,

$$\begin{aligned}\|e_2\|_{L^2} &\le (8+\ln(n\varepsilon^{-2}))\tau^2 \int_0^1 \|B(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))\|_{L^2}ds \\ &\le (8+\ln(n\varepsilon^{-2}))^2\tau^2 \int_0^1 \|\Phi_B^{s\tau}(e^{i\tau\Delta}v_0)\|_{L^2}ds \\ &= (8+\ln(n\varepsilon^{-2}))^2\|v_0\|_{L^2}\tau^2. \end{aligned} \tag{4.31}$$

Combining (4.28), (4.30) and (4.31), when $\varepsilon \le \widetilde{\varepsilon}_0 = \min\{\varepsilon_2, \varepsilon_3\} = \varepsilon_2$, we have

$$\begin{aligned}\|\Psi^\tau(v_0) - \Phi^\tau(v_0)\|_{L^2} &\le \tau^2\|v_0\|_{H^2}\Big[c_1 + c_2\ln(n\varepsilon^{-2}) + c_3(\ln(n\varepsilon^{-2}))^2 \\ &\quad + \frac{12cn^2}{\varepsilon}\|v_0\|_{H^2}\Big] \\ &\le \tau^2\|v_0\|_{H^2}\left[\frac{c_1}{\varepsilon} + \frac{C_2 n^{1/2}}{\varepsilon} + \frac{12cn^2}{\varepsilon}\|v_0\|_{H^2}\right] \\ &\le C(n, \|v_0\|_{H^2})\frac{\tau^2}{\varepsilon}, \end{aligned}$$

where we have employed the inequalities $\ln(x) \le Cx^{1/2}$ and $\ln(x) \le Cx^{1/4}$ for $x \in [1,\infty)$. Hence (4.23) is established.

Similarly the stability can be yielded by (4.6):

$$\|\Phi^\tau(v) - \Phi^\tau(w)\|_{L^2} \le (1+4n\tau)\|\Phi_A^\tau(v-w)\|_{L^2} = (1+4n\tau)\|v-w\|_{L^2}, \tag{4.32}$$

for $v, w \in L^2(\Omega)$. Denote $\varepsilon_0 = \frac{\sqrt{n}/c}{\|u\|_{L^\infty([0,T];H^1)}}$, then by applying similar arguments in the proof of Theorem 4.1, we can get the error estimate (4.22). $\square$

Remark 4.3. For $d = 2, 3$, the error estimate (4.22) can be established with ε_0 depending on n, τ_0 and $\|u^\varepsilon\|_{L^\infty([0,T];H^2(\Omega))}$ by noticing that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ for $d = 2, 3$.

Remark 4.4 (Strang splitting). When considering a Strang splitting,

$$u^{\varepsilon,k+1} = \Phi_B^{\tau/2}(\Phi_A^\tau(\Phi_B^{\tau/2}(u^{\varepsilon,k}))), \quad \text{or} \quad u^{\varepsilon,k+1} = \Phi_A^{\tau/2}(\Phi_B^\tau(\Phi_A^{\tau/2}(u^{\varepsilon,k}))), \tag{4.33}$$

by applying similar but more intricate arguments as above, we can prove the error bound

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \le C(n, \tau_0, T, \|u^\varepsilon\|_{L^\infty([0,T];H^4(\Omega))})\,\frac{\tau^2}{\varepsilon^3},$$

under the assumption that $u^\varepsilon \in L^\infty([0,T]; H^4(\Omega))$.

Remark 4.5. In view of Theorems 3.1, 4.1 and 4.2 rely on a regularity that we know is available. On the other hand, the regularity assumed in the above remark on Strang splitting is unclear in general, in the sense that we don't know how to bound u^ε in $L^\infty([0,T];H^4(\Omega))$.

5. Numerical Results

In this section, we first test the convergence rate of the local energy regularized model (3.2) and compare it with the other two (1.6) and (1.7). We then test the order of accuracy of the regularized LTSP schemes (4.7) and (4.20) and Strang splitting (STSP) scheme (4.33). To simplify the presentation, we unify the regularized models (1.6), (1.7) and (3.2) as follows:

$$\begin{cases} i\partial_t u^\varepsilon(\boldsymbol{x},t) + \Delta u^\varepsilon(\boldsymbol{x},t) = \lambda u^\varepsilon(\boldsymbol{x},t) f^\varepsilon_{\mathrm{reg}}(|u^\varepsilon(\boldsymbol{x},t)|^2), & \boldsymbol{x}\in\Omega,\ \ t>0, \\ u^\varepsilon(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x}\in\overline{\Omega}. \end{cases} \tag{5.1}$$

With the regularized nonlinearity $f^\varepsilon_{\mathrm{reg}}(\rho)$ being chosen as $\widetilde{f}^\varepsilon$, $\widehat{f}^\varepsilon$ and f^ε_n, (5.1) corresponds to the regularized models (1.6), (1.7) and (3.2), respectively. In practical computation, we impose periodic boundary condition on Ω and employ the standard Fourier pseudo-spectral method[4,6,8] for spatial discretization. The details are omitted here for brevity.

Hereafter, unless specified, we consider the following Gaussian initial data in d-dimension ($d = 1, 2$), i.e. $u_0(\boldsymbol{x})$ is chosen as

$$u_0(\boldsymbol{x}) = b_d\, e^{i\boldsymbol{x}\cdot\boldsymbol{v} + \frac{\lambda}{2}|\boldsymbol{x}|^2}, \quad \boldsymbol{x}\in\mathbb{R}^d. \tag{5.2}$$

In this case, the LogSE (1.1) admits the moving Gausson solution

$$u(\boldsymbol{x},t) = b_d\, e^{i(\boldsymbol{x}\cdot\boldsymbol{v} - (a_d + |\boldsymbol{v}|^2)t) + \frac{\lambda}{2}|\boldsymbol{x}-2\boldsymbol{v}t|^2}, \quad \boldsymbol{x}\in\mathbb{R}^d, \quad t\ge 0, \tag{5.3}$$

with $a_d = -\lambda\,(d - \ln|b_d|^2)$. In this paper, we let $\lambda = -1$, $b_d = 1/\sqrt[4]{-\lambda\pi}$ and choose $\Omega = [-16, 16]^d$. Moreover, we fix $v = 1$ and $\boldsymbol{v} = (1,1)^T$ as well as take the mesh size as $h = 1/64$ and $h_x = h_y = 1/16$ for $d = 1$ and 2, respectively. To quantify the numerical errors, we define the following error functions:

$$\begin{aligned} &\check{e}^\varepsilon_\rho(t_k) := \rho(\cdot,t_k) - \rho^\varepsilon(\cdot,t_k) = |u(\cdot,t_k)|^2 - |u^\varepsilon(\cdot,t_k)|^2, \\ &\check{e}^\varepsilon(t_k) := u(\cdot,t_k) - u^\varepsilon(\cdot,t_k), \qquad \check{\check{e}}^\varepsilon(t_k) := u(\cdot,t_k) - u^{\varepsilon,k}, \\ &e^\varepsilon(t_k) := u^\varepsilon(\cdot,t_k) - u^{\varepsilon,k}, \quad e^\varepsilon_E := |E(u_0) - E^\varepsilon_{\mathrm{reg}}(u_0)|. \end{aligned} \tag{5.4}$$

Here, u and u^ε are the exact solutions of the LogSE (1.1) and RLogSE (5.1), respectively, while $u^{\varepsilon,k}$ is the numerical solution of the RLogSE (5.1) obtained by LTSP (4.7) (or (4.20)) or STSP (4.33). The "exact" solution u^ε is obtained numerically by STSP (4.33) with a very small time step, e.g. $\tau = 10^{-5}$. The energy is obtained by the trapezoidal rule for approximating the integrals in the energy (1.4), (3.1), (3.9) and (3.10).

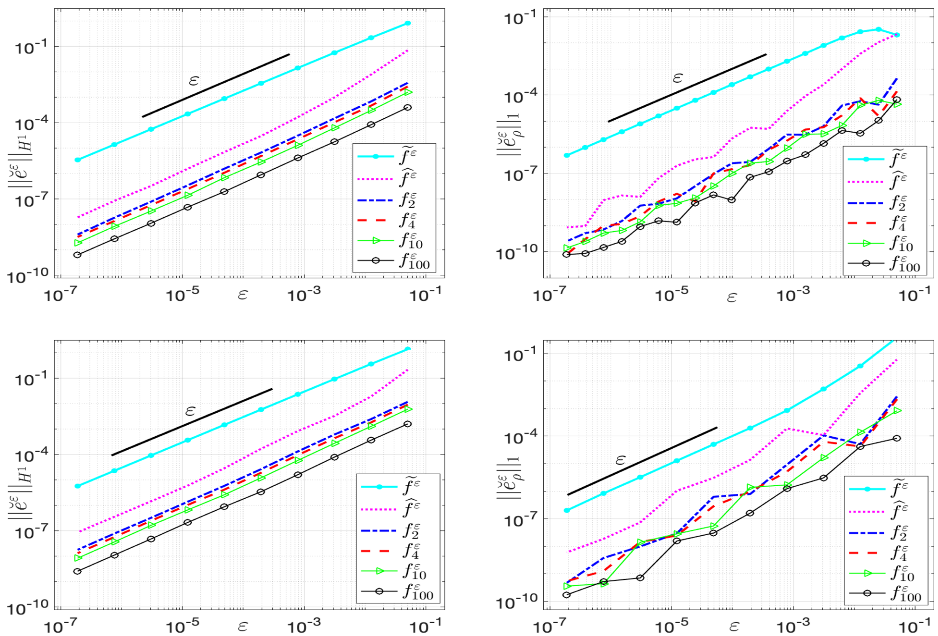

Fig. 6. (Color online) Convergence of the RLogSE (5.1) with various regularized nonlinearities $f^\varepsilon_{\rm reg}$ to the LogSE (1.1), i.e. the error $\|\check{e}^\varepsilon(t)\|_{H^1}$ and $\|\check{e}^\varepsilon_\rho(t)\|_1$ versus the regularization parameter ε at $t = 3$ for $d = 1$ (upper) and $t = 2$ for $d = 2$ (lower).

5.1. *Convergence rate of the regularized model*

Here, we consider the error between the solutions of the RLogSE (5.1) and the LogSE (1.1). For various regularized models (i.e. different choices of regularized nonlinearity $f^\varepsilon_{\rm reg}$ in Eq. (5.1)), Fig. 6 shows $\|\check{e}^\varepsilon(t)\|_{H^1}$ and $\|\check{e}^\varepsilon_\rho(t)\|_1$ at $t = 3$ and $t = 2$, respectively, for $d = 1$ and 2, while Fig. 7 depicts e^ε_E versus ε. The results are similar when $\check{e}^\varepsilon(t)$ is measured by L^2- or L^∞-norm.

From these figures and additional similar numerical results not shown here for brevity, we could clearly see: (i) The solution of the RLogSE (5.1) converges linearly to that of the LogSE (1.1) in terms of ε for all the three types of regularized models. Moreover, the regularized energy $\widetilde{E}^\varepsilon$ converges linearly to the original energy E in terms of ε, while $\widehat{E}^\varepsilon$ and E^ε_n (for any $n \geq 2$) converges quadratically. These results confirm the theoretical results from Secs. 3.2 and 3.3. (ii) In L^1-norm, the density ρ^ε of the solution of the RLogSE with regularized nonlinearity $\widetilde{f}^\varepsilon$ converges linearly to that of the LogSE (1.1) in terms of ε, while the convergence rate is not clear for those of RLogSE with other regularized nonlinearities. Generally, for fixed ε, the errors of the densities measured in L^1-norms are smaller than those of wave functions (measured in L^2, H^1 or L^∞-norm). (iii) For any fixed $\varepsilon > 0$, the proposed LER (i.e. $f^\varepsilon_{\rm reg} = f^\varepsilon_n$) outperforms the other two (i.e. $f^\varepsilon_{\rm reg} = \widehat{f}^\varepsilon$ and $f^\varepsilon_{\rm reg} = \widetilde{f}^\varepsilon$) in the sense that its corresponding errors in wave function and total energy are smaller.

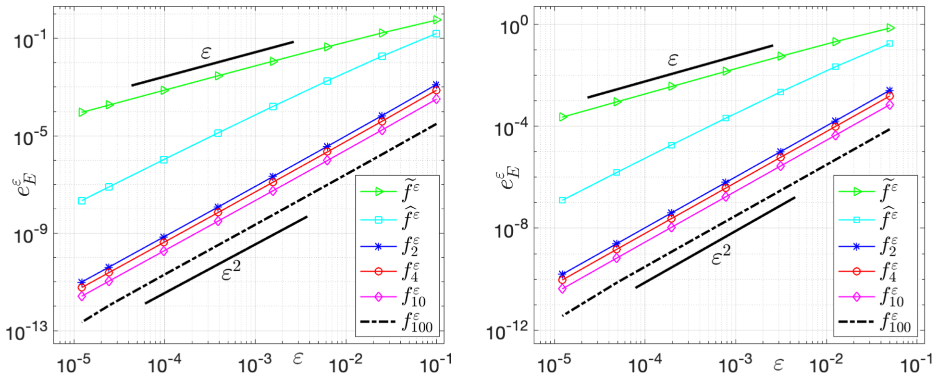

Fig. 7. (Color online) Convergence of the RLogSE (5.1) with various regularized nonlinearities $f_{\rm reg}^\varepsilon$ to the LogSE (1.1): the energy error $e_E^\varepsilon(t)$ (5.4) at $t = 3$ for $d = 1$ (left) and $t = 2$ for $d = 2$ (right).

The larger the order (i.e. n) of the energy-regularization is chosen, the smaller the difference between the solutions of the ERLogSE (3.2) and LogSE is obtained.

5.2. *Convergence rate of the time-splitting spectral method*

Here, we investigate the model RLogSE (5.1) with $f_{\rm reg}^\varepsilon = f_n^\varepsilon$, i.e. the ERLogSE (3.2). We will test the convergence rate of type-1 LTSP (4.7) and type-2 LTSP (4.20) and the STSP (4.33) to the ERLogSE (3.2) or the LogSE (1.1) in terms of the time step τ for fixed $\varepsilon \in (0, 1)$. Figure 8 shows the errors $\|e^\varepsilon(3)\|_{H^1}$ versus time step τ for f_2^ε and f_4^ε. In addition, Table 1 displays $\|\breve{e}^\varepsilon(3)\|$ versus ε and τ for f_2^ε.

From Fig. 8, Table 1 and additional similar results not shown here for brevity, we can observe that: (i) In H^1 norm, for any fixed $\varepsilon \in (0, 1)$ and $n \geq 2$, the LTSP scheme converges linearly while the STSP scheme converges quadratically

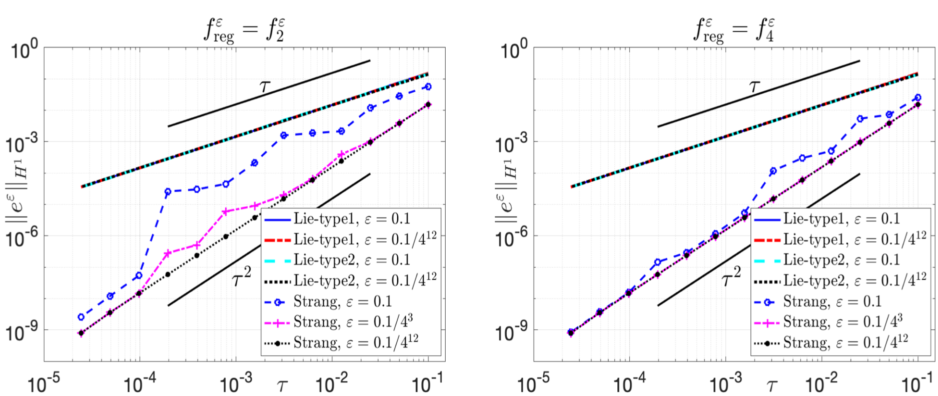

Fig. 8. (Color online) Convergence of the type-1 LTSP (4.7) and type-2 LTSP (4.20) as well as the STSP (4.33) to the ERLogSE (3.2) with regularized nonlinearity f_2^ε (left) and f_4^ε (right), i.e. errors $\|e^\varepsilon(3)\|_{H_1}$ versus τ for various ε.

Table 1. Convergence of the STSP (4.33) (via solving the ERLogSE (3.2) with f_2^ε) to the LogSE (1.1), i.e. $\|\breve{e}^\varepsilon(3)\|$ for different ε and τ.

	$\tau = 0.1$	$\tau/2$	$\tau/2^2$	$\tau/2^3$	$\tau/2^4$	$\tau/2^5$	$\tau/2^6$	$\tau/2^7$	$\tau/2^8$	$\tau/2^9$
$\varepsilon = 0.025$	7.98E-3	**2.13E-3**	8.86E-4	7.28E-4	7.14E-4	7.12E-4	7.12E-4	7.12E-4	7.12E-4	7.12E-4
rate	—	**1.91**	1.27	0.28	0.03	0.00	0.00	0.00	0.00	0.00
$\varepsilon/4$	7.77E-3	1.96E-3	**5.02E-4**	1.67E-4	1.12E-4	1.08E-4	1.08E-4	1.08E-4	1.08E-4	1.08E-4
rate	—	1.99	**1.97**	1.59	0.57	0.06	0.01	0.00	0.00	0.00
$\varepsilon/4^2$	7.76E-3	1.95E-3	4.88E-4	**1.25E-4**	3.81E-5	2.40E-5	2.28E-5	2.27E-5	2.27E-5	2.27E-5
rate	—	2.00	2.00	**1.97**	1.71	0.67	0.07	0.01	0.00	0.00
$\varepsilon/4^3$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	**3.08E-5**	8.95E-6	5.09E-6	4.74E-6	4.72E-6	4.71E-6
rate	—	2.00	2.00	2.00	**1.98**	1.78	0.82	0.10	0.01	0.00
$\varepsilon/4^4$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	3.04E-5	**7.66E-6**	2.092E-6	9.93E-7	8.80E-7	8.72E-7
rate	—	2.00	2.00	2.00	2.00	**1.99**	1.87	1.08	0.18	0.01
$\varepsilon/4^5$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	3.04E-5	7.61E-6	**1.92E-6**	5.26E-7	2.54E-7	2.27E-7
rate	—	2.00	2.00	2.00	2.00	2.00	**1.99**	1.87	1.05	0.16
$\varepsilon/4^6$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	3.04E-5	7.61E-6	1.90E-6	**4.78E-7**	1.27E-7	5.36E-8
rate	—	2.00	2.00	2.00	2.00	2.00	2.00	**1.99**	1.91	1.25
$\varepsilon/4^7$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	3.04E-5	7.61E-6	1.90E-6	4.76E-7	**1.19E-7**	3.13E-8
rate	—	2.00	2.00	2.00	2.00	2.00	2.00	2.00	**2.00**	1.93
$\varepsilon/4^8$	7.76E-3	1.95E-3	4.87E-4	1.22E-4	3.04E-5	7.61E-6	1.90E-6	4.76E-7	1.19E-7	**2.98E-8**
rate	—	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	**2.00**

when $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$. (ii) For any f_n^ε with $n \geq 2$, the STSP converges quadratically to the LogSE (1.1) only when ε is sufficiently small, i.e. $\varepsilon \lesssim \tau^2$ (cf. each row in the lower triangle below the diagonal in bold letter in Table 1). (iii) When τ is sufficiently small, i.e. $\tau^2 \lesssim \varepsilon$, the ERLogSE (3.2) converges linearly at $O(\varepsilon)$ to the LogSE (1.1) (cf. each column in the upper triangle above the diagonal in bold letter in Table 1). (iv) The numerical results are similar for other f_n^ε with $n \geq 3$ and when the errors are measured in L^∞- and L^2-norm, which confirm the theoretical conclusion in Theorem 4.2 and Remark 4.4.

5.3. *Application for interaction of 2D Gaussons*

In this section, we apply the STSP method to investigate the interaction of Gaussons in dimension 2. To this end, we fix $n = 4$, $\varepsilon = 10^{-12}$, $\tau = 0.001$, $h_x = h_y = 1/16$, $\Omega = [-16, 16]^2$ for *Cases* 1 and 2 while $\Omega = [-48, 48]^2$ for *Case* 3. The initial data is chosen as

$$u_0(\boldsymbol{x}) = b_1 e^{i\boldsymbol{x}\cdot\boldsymbol{v}_1 + \frac{\lambda}{2}|\boldsymbol{x}-\boldsymbol{x}_1^0|^2} + b_2 e^{i\boldsymbol{x}\cdot\boldsymbol{v}_2 + \frac{\lambda}{2}|\boldsymbol{x}-\boldsymbol{x}_2^0|^2}, \tag{5.5}$$

where b_j, $\boldsymbol{v}_j$ and $\boldsymbol{x}_j^0$ ($j = 1, 2$) are real constant vectors, i.e. the initial data is the sum of two Gaussons (5.3) with velocity $\boldsymbol{v}_j$ and initial location $\boldsymbol{x}_j^0$. Here, we consider the following cases:

(i) $b_1 = b_2 = \frac{1}{\sqrt[4]{\pi}}$, $\boldsymbol{v}_1 = \boldsymbol{v_2} = (0,0)^T$, $\boldsymbol{x}_1^0 = -\boldsymbol{x}_2^0 = (-2,0)^T$;

(ii) $b_1 = 1.5\, b_2 = \frac{1}{\sqrt[4]{\pi}}$, $\boldsymbol{v}_1 = (-0.15,0)^T$, $\boldsymbol{v_2} = \boldsymbol{x}_1^0 = (0,0)^T$, $\boldsymbol{x}_2^0 = (5,0)^T$;

(iii) $b_1 = b_2 = \frac{1}{\sqrt[4]{\pi}}$, $\boldsymbol{v}_1 = (0,0)^T$, $\boldsymbol{v_2} = (0,0.85)^T$, $\boldsymbol{x}_1^0 = -\boldsymbol{x}_2^0 = (-2,0)^T$.

Figure 9 shows the contour plots of $|u^\varepsilon(x,y,t)|^2$ at different time as well as the evolution of $\sqrt{|u^\varepsilon(x,0,t)|}$ for *Cases* (i) and (ii). While Fig. 10 illustrates that for *Case* (iii). From these figures we clearly see that: (1) Even for two static Gaussons,

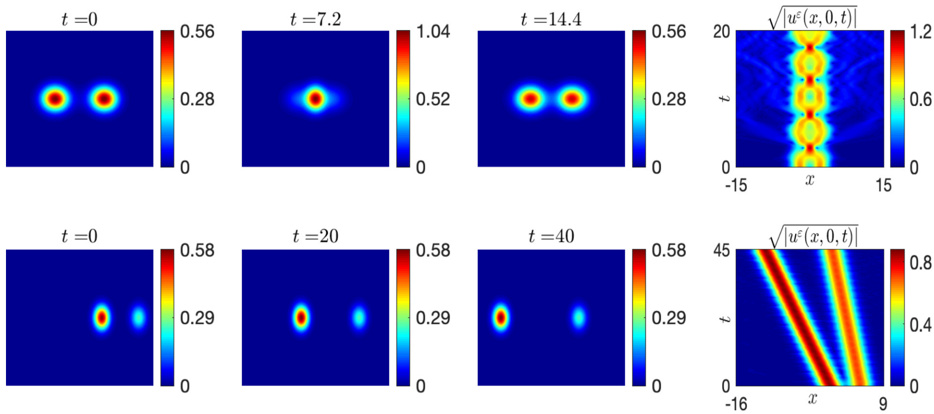

Fig. 9. (Color online) Plots of $|u^\varepsilon(x, y, t)|^2$ at different times (first three column) and contour plot of $|u^\varepsilon(x, 0, t)|^2$ (last column) for *Case* (i) (Upper) in region $[-6, 6]^2$ and *Case* (ii) (Lower) in region $[-13, 7] \times [-6, 6]$.

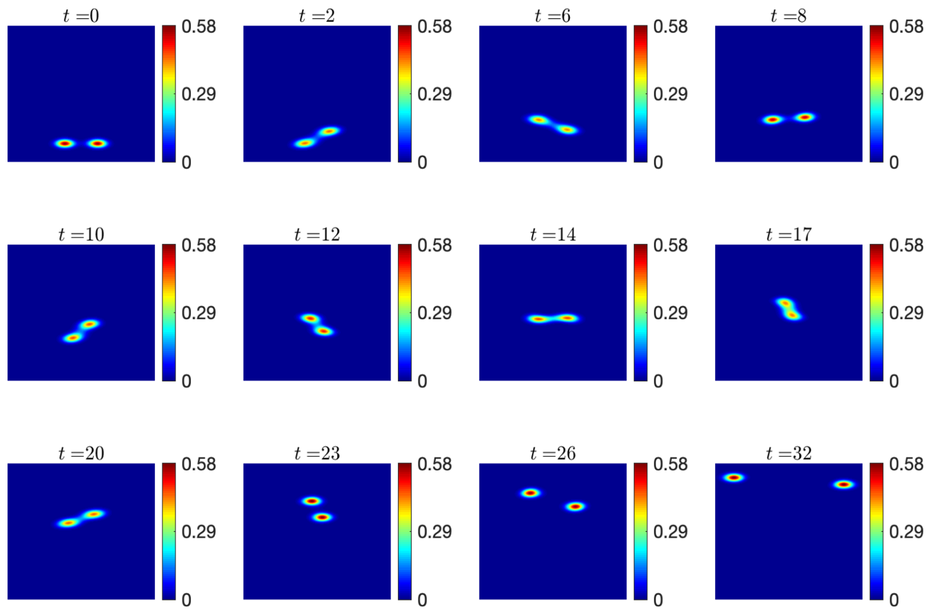

Fig. 10. (Color online) Plots of $|u^\varepsilon(x, y, t)|^2$ at different times for *Case* (iii) in region $[-9, 9] \times [-5, 32]$.

if they stay close enough, they will contact and undergo attractive interactions. They will collide and stick together shortly then separate again. The Gaussons will swing like a pendulum and small solitary waves are emitted outward during the interaction (cf. Fig. 9). This dynamics phenomena is similar to that in 1D case.[4] (2) For Case (ii), the two Gaussons also undergo attractive interactions. The slowly moving Gausson will drag its nearby static Gausson to move in the same direction (cf. Fig. 9), which is also similar to that in 1D case.[4] (3) For two Gaussons (one static and the other moving) staying close enough, if the moving Gausson move perpendicular to the line connecting the two Gaussons, the static Gausson will be dragged to move and the direction of the moving Gausson will be altered. The two Gaussons will rotate with each other and gradually drift away, which is similar to the dynamics of a vortex pair in the cubic Schrödinger equation.[10]

6. Conclusion

We proposed a new systematic LER approach to overcome the singularity of the nonlinearity in the LogSE. With a small regularized parameter $0 < \varepsilon \ll 1$, in contrast to the existing ones that directly regularize the logarithmic nonlinearity, we regularized locally the interaction energy density in the energy functional of the LogSE. The Hamiltonian flow of the new regularized energy then yields an ERLogSE. Linear and quadratic convergence in terms of ε was established between the solutions, and between the conserved total energy of ERLogSE and LogSE, respectively. Then we presented and analyzed time-splitting schemes to solve the ERLogSE. The classical first order of convergence was obtained both theoretically and numerically for the LTSP scheme. Numerical results suggest that the error bounds of splitting schemes to the LogSE clearly depend on the time step τ and mesh size h as well as the small regularized parameter ε. Our numerical results confirm the error bounds and indicate that the ERLogSE model outperforms the other existing ones in accuracy.

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